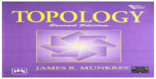
MMA 32 Topology



Dr S. Srinivasan

Assistant Professor,
Department of Mathematics,
Periyar Arts College,
Cuddalore - 1, Tamil nadu

Email: smrail@gmail.com
Cell: 7010939424

Definition 1.

A **topology** on a set X is a collection τ of subsets of X having the following properties:

(1) ϕ and X are in τ.

(2) The union of the elements of any sub collection of τ is in τ.

i.e., if $\{U_\alpha\}_{\alpha \in A} \subset \tau$ then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

(3) The intersection of the elements of any finite sub collection of τ is in τ.

i.e., if $U_1, U_2, \ldots, U_n \in \tau$ then $\bigcap_{i=1}^{n} U_i \in \tau$.

Definition 2.

A set X for which a topology τ has been specified is

called a **topological space**.

Note:

1. An ordered pair (X, τ) consisting a set and a topology τ on X.

2. A subset of X which is in τ is called an **open set**.

   i.e., if $U \in \tau \Rightarrow U$ is an open set of X.

Example 1. Let $X = \{a, b, c\}$.

Here We list 9 topologies on X. There are

(1) The **trivial topology** $\tau_1 = \{\phi, X\}$.

(2) $\tau_2 = \{\phi, \{a\}, X\}$.
(3) $\tau_3 = \{\phi, \{a, b\}, X\}$.
(4) $\tau_4 = \{\phi, \{a\}, \{a, b\}, X\}$.
(5) $\tau_5 = \{\phi, \{a, b\}, \{c\}, X\}$.
(6) $\tau_6 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.
(7) $\tau_7 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$.
(8) $\tau_8 = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$.

(9) The **discrete topology** $\tau_9 = P(X)$ (power set with 8 elements).

Example. Let $X = \{a, b, c\}$.

Here are some collections of subsets of X that are **not topologies**.

(1) $\{\{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ does not contain ϕ and X.

(2) $\{\phi, \{a\}, \{b\}, X\}$ is not closed under union.

(3) $\{\phi, \{a, b\}, \{a, c\}, X\}$ is not closed under finite intersection.

Example 2. Let X be a set.

τ_f be the collection of all subsets U of X such that

$X - U = X \backslash U = \{x \in X \mid x \notin U\}$ is either finite or all of X.

Then τ_f is a topology on X, called the **finite complement topology**.

Example 3. Let X be a set.

τ_c be the collection of all subsets U of X such that

$X \backslash U$ is either countable or all of X.

Then τ_c is a topology on X.

Definition 3.

Suppose that τ and τ' are two topologies on a given set X.

If $\tau' \supset \tau$ then τ' is **finer** then τ.

If τ' properly contains τ then τ' is **strictly finer** than τ.

We also say that τ is **coarser** then τ', or τ is **strictly coarser** then τ',

respectively.

We say τ is **comparable** with τ' if either $\tau' \supset \tau$ or $\tau \supset \tau'$

Note:

1. If τ' is finer than τ then τ' has more open sets than τ.

2. The trivial topology is coarser than any other topology, and

   the discrete topology is finer than any other topology.

1. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1. Compare them, i.e.,for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

2. If $\{\tau_\alpha\}$ is a family of topologies on X, show that $\bigcap \tau_\alpha$ is a topology on X. Is $\bigcup \tau_\alpha$ a topology on X?

3. If $X = \{a, b, c\}$, let $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing τ_1 and τ_2, and the largest topology contained in τ_1 and τ_2.

4. Let $\{\tau_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections τ_α, and a unique largest topology contained in all τ_α.

5. Let $X = \{a, b, c, d, e\}$. Determine whether or not each of the following classes of subsets of X is a topology on X.
   (i) $\tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$
   (ii) $\tau_2 = \{\phi, X, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$
   (iii) $\tau_3 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$

Definition 1.

Let X be a set.

A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X

(called basis elements) such that

(1) For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$

such that $x \in B$.

(2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is a $B_3 \in \mathcal{B}$

such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$

$\mathcal{T}$ generated by $\mathcal{B}$

Definition 2.

The topology $\mathcal{T}$ generated by $\mathcal{B}$ is defined as follows:

A subset U of X is said to be open in X (i.e.,$U \in \mathcal{T}$)

if for each $x \in U$ there is a basis element $B \in \mathcal{B}$ such that

$x \in B$ and $B \subset U$.

Note :

1. Therefore each basis element is in $\mathcal{T}$

2. In fact, the topology generated by basis $\mathcal{B}$ is a topology.

Example 1. A basis for the standard topology on $\mathbb{R}^2$

is given by the set of all circular regions in $\mathbb{R}^2$:

$\mathcal{B} = \{B((x_0, y_0), r) \mid r > 0\}$ where

$$B((x_0, y_0), r) = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

Example 2. If X is any set.

$$\mathcal{B} = \{\{x\} \mid x \in X\}$$

is a basis for the discrete topology on X.

Theorem A.

Let X be a set and

$\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X.

Define $\mathcal{T} = \{U \subset X \mid x \in U \text{ implies } x \in B \subset U \text{ for some } B \in \mathcal{B}\}$.

the *topology generated by* $\mathcal{B}$.

Then $\mathcal{T}$ is in fact a topology on X.

Lemma 1. Let X be a set.

Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X.

Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof.

By Theorem A above, all elements of $\mathcal{B}$ are open and so in $\mathcal{T}$.

Since $\mathcal{T}$ is a topology, then by part (2) of the definition,

any union of elements of $\mathcal{B}$ are in $\mathcal{T}$.

$\Rightarrow \mathcal{T}$ contains all unions of elements of $\mathcal{B}$.

Conversely, given $U \in \mathcal{T}$.

For each $x \in U$.

Choose $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$ ($\mathcal{T}$ generated by $\mathcal{B}$).

Then $U = \bigcup_{x \in U} B_x$.

i.e., U equals a union of elements of $\mathcal{B}$.

Since U is an arbitrary element of $\mathcal{T}$,

then all elements of $\mathcal{T}$ are unions of elements of $\mathcal{B}$.

Lemma 2. Let $(X, \mathcal{T})$ be a topological space.

Suppose that $\mathcal{C}$ is a collection of open sets of X such that

for each open subset $U \subset X$ and each $x \in U$, there is an element

$C \in \mathcal{C}$ such that $x \in C \subset U$.

Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$ on X.

Proof. First we show that $\mathcal{C}$ is a basis.

(i) By the definition of basis, for $x \in X$.

(since X itself is an open set)

Then (by hypothesis) there is an element $C \in \mathcal{C}$ such that $x \in C \subset X$.

(ii) For the second part of the definition of basis.

Let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$.

Since C_1 and C_2 are open then $C_1 \cap C_2$ is open.

Then by hypothesis,there is an element $C_3 \in \mathcal{C}$ such that

$x \in C_3 \subset C_1 \cap C_2$.

Thus $\mathcal{C}$ is a basis for a topology on X.

Let $\mathcal{T}'$ be the topology on X generated by $\mathcal{C}$.

To prove that $\mathcal{T} = \mathcal{T}'$.

First, Let $U \in \mathcal{T}$ and $x \in U$.

Since $\mathcal{C}$ is a basis for topology $\mathcal{T}$,

$\Rightarrow$ there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$.

i.e., $U \in \mathcal{T}'$. (by the def of *topology generated by* $\mathcal{C}$)

Hence $\mathcal{T} \subset \mathcal{T}'$.

Conversely,

If W belongs to $\mathcal{T}'$.

Then W is a union of elements of $\mathcal{C}$. (by Lemma 1)

Now each element of $\mathcal{C}$ is an element of $\mathcal{T}$.

(by the definition of *topology generated by*)

(and a union of open sets is open)

$\Rightarrow W$ belongs to $\mathcal{T}$.

That is, $\mathcal{T}' \subset \mathcal{T}$.

Therefore, $\mathcal{T} = \mathcal{T}'$.

Lemma 3. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof. (2) $\Rightarrow$ (1)

Given $U \in \mathcal{T}$, let $x \in U$.

Since $\mathcal{B}$ generates $\mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.

By hypothesis (2), there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

$\Rightarrow x \in B' \subset U$.

$\Rightarrow U \in \mathcal{T}'$. (By the definition of *topology generated by* $\mathcal{B}'$.)

$\Rightarrow \mathcal{T} \subset \mathcal{T}'$.

(1) $\Rightarrow$ (2)

Let $x \in X$ and $B \in \mathcal{B}$ where $x \in B$.

Since $\mathcal{B}$ generates $\mathcal{T}$, then $B \in \mathcal{T}$.

By hypothesis (1), $\mathcal{T} \subset \mathcal{T}'$ and so $B \in \mathcal{T}'$.

Since $\mathcal{T}'$ is generated by $\mathcal{B}'$.

Then there is (by definition) $B' \in \mathcal{B}$ such that $x \in B' \subset B$.

Definition 3.

Let $\mathcal{B}$ be the set of all open intervals in the real line:

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}, \text{ where}$$

$$(a, b) = \{x \mid a < x < b\}$$

The topology generated by $\mathcal{B}$ is the standard topology on $\mathbb{R}$.

Definition 4.

Let $\mathcal{B}'$ be the set of all half open intervals.

$$\mathcal{B}' = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}, \text{ where}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

The topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$.

It is denoted by $\mathbb{R}_\ell$.

Definition 5.

Let $K = \{1/n \mid n \in N\}$.

$$\mathcal{B}'' = \{(a, b) \mid a, b \in \mathbb{R}, a < b\} \cup \{(a, b) - K \mid a, b \in \mathbb{R}, a < b\}.$$

The topology generated by $\mathcal{B}''$ is the K-topology on $\mathbb{R}$.

It is denoted by $\mathbb{R}_K$.

Lemma 4. The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are each strictly finer

than the standard topology on $\mathbb{R}$.

But are not comparable with one another.

Proof.

Let $\mathcal{T}, \mathcal{T}'$, and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_\ell$ and $\mathbb{R}_K$ respectively.

Given a basis element (a, b) for $\mathcal{T}$ and $x \in (a, b)$,

Then the basis element $[x, b) \in \mathcal{T}'$ contains x and lies in (a, b)

i.e., $x \in [x, b) \subset (a, b)$.

On the other hand, given basis element $[x, d) \in \mathcal{T}'$, there is no

open interval (a, b) containing x which is a subset of $[x, d)$.

By Lemma 3.(2) $\Rightarrow \mathcal{T}'$ is strictly finer than $\mathcal{T}$.

Given a basis element (a, b) for $\mathcal{T}$ and $x \in (a, b)$.

Then this same basis element $(a, b) \in \mathcal{T}''$ contains x.

Which satisfies $x \in (a, b) \subset (a, b)$.

On the other hand, given the basis element $C = (-1, 1) - K$ for $\mathcal{T}''$.

Then the point $0 \in C$.

But there is no open interval (a, b) containing 0

which is a subset of C. (For example $(\frac{-1}{2}, \frac{1}{2}) \notin C$)

By Lemma 3.(2) $\Rightarrow \mathcal{T}''$ is strictly finer than $\mathcal{T}$.

Show that topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

Let $\mathcal{T}_\ell$ and $\mathcal{T}_K$ be the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$, respectively.

It suffices to show that neither of the topologies is

finer than the other.

i.e., to prove $\mathcal{T}_\ell \not\subset \mathcal{T}_K$ and $\mathcal{T}_K \not\subset \mathcal{T}_\ell$.

Given $x \in \mathbb{R}$ where $a < x < b$ is contained in the basis

element $[x, b)$ of $\mathbb{R}_\ell$.

However, every basis element of $\mathbb{R}_K$ is an open interval

(in some cases, minus the set K).

There is no open interval (a, b) that contains x and

is contained in $[x, b)$ because $a < x$.

By Lemma 3(2), $\mathcal{T}_K$ is not finer than $\mathcal{T}_\ell$.

Conversely, 0 is contained in the basis element $(-1, 1) - K$ of $\mathcal{T}_K$.

Any basis element $[a, b)$ of $\mathcal{T}_\ell$ contains 0, where $a < 0$ and $b > 0$.

But this basis element cannot be contained in $(-1, 1) - K$.

Given $b > 0$, let $k \in \mathbb{N}$ where $k > 1/b$.

It follows that $0 < 1/k < b$, $\Rightarrow 1/k \in [a, b)$.

But $1/k \notin (-1, 1) - K$.

Again by Lemma 3(2), $\mathcal{T}_\ell$ is not finer than $\mathcal{T}_K$.

Hence $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are not comparable.

Definition 5.

A subbasis $\mathcal{S}$ for a topology on set X is a collection of subsets of X whose union equals X.

The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Example

Example 1.

Observe that every open interval (a, b) in the line $\mathbb{R}$ is the intersection of two infinite open intervals (a, ∞) and $(-\infty, b)$

$$(a, b) = (-\infty, b) \cap (a, \infty).$$

But the open intervals form a base for the usual topology on $\mathbb{R}$.

Hence the class $\mathcal{S}$ of all infinite open intervals is a subbase for $\mathbb{R}$.

Example 2.

If $X = \{a, b, c, d\}$ and $\mathcal{S} = \{\{a, b, c\}, \{b, c, d\}\}$ then the topology generated by $\mathcal{S}$ is

$\tau = \{\phi, \{a, b, c\}, \{b, c, d\}, \{b, c\}, \{a, b, c, d\}\}$.

Theorem B.

Let $\mathcal{S}$ be a subbasis for a topology on X.

Define $\mathcal{T}$ to be all unions of finite intersections of elements of $\mathcal{S}$.

Then $\mathcal{T}$ is a topology on X.

Definition 1. *Intervals*

Let$X$ be a set with a simple order relation $<$.

The following sets are intervals in X :

$(a, b) = \{x \in X | a < x < b\}$ (open intervals)

$(a, b] = \{x \in X | a < x \leq b\}$ (half-open intervals)

$[a, b) = \{x \in X | a \leq x < b\}$ (half-open intervals)

$[a, b] = \{x \in X | a \leq x \leq b\}$ (closed intervals).

Definition 2.

Let X be a set with a simple order relation and assume X hax

more than one element.

Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$ where a_0 is the least element of X.

(3) All intervals of the form $(a, b_0]$ where b_0 is the greatest element of X.

The collection $\mathcal{B}$ is a basis for a topology on X called the order topology.

Example 1.

The standard topology on $\mathbb{R}$ is the order topology based on the

usual *less than* order on $\mathbb{R}$.

Example 2.

We can put a simple order relation on $\mathbb{R}^2$ as follows:

$(a, b) < (c, d)$ if either

(1) $a < c$, or

(2) $a = c$ and $b < d$.

This is often called the lexicographic ordering

Definition 3.

If X is a set with a simple order relation $<$, and $a \in X$ then there are four subsets of X, called rays determined by a.

They are the following:

$$(a, \infty) = \{x \in X | x > a\} \qquad (open\ rays)$$

$$(-\infty, a) = \{x \in X | x < a\} \qquad (open\ rays)$$

$$[a, \infty) = \{x \in X | x \geq a\} \qquad (closed\ rays)$$

$$(-\infty, a] = \{x \in X | x \leq a\}. \qquad (closed\ rays)$$

1.4 The Product Topology on $X \times Y$

If X and Y are topological spaces, then there is a

natural topology on the Cartesian product.

$$X \times Y = \{(x, y) | x \in X, y \in Y\}. \qquad \text{(product topology)}$$

Definition 1. Basis

Let X and Y be topological spaces.

The product topology on set $X \times Y$ is the topology having as basis

the collection $\mathcal{B}$ of all sets of the form $U \times V$,

where U is an open subset of X and V is an open subset of Y.

Theorem 1. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$D = \{B \times C | B \in \mathcal{B} \& C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof.

Let $W \in X \times Y$ be an open set.

Let $(x, y) \in W$.

By the definition of product topology, there is a basis element $U \times V$, where U is open in X and V is open in Y, such that $(x, y) \in U \times V \subset W$.

$\Rightarrow x \in U$ and $y \in V$.

Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, respectively.

Then there are open sets $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subset U$

and $y \in C \subset V$.

Notice that $B \times C$ is an element of the basis for the product topology

and so open and $B \times C \in D$.

That is, $(x, y) \in B \times C \subset W$ where $B \times C \in D$.

By Lemma 2, D is a basis for the product topology.

Theorem 2. The set

$$S = \{\pi_1^{-1}(U) | U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let τ denote the product topology on $X \times Y$.

Let τ' be the topology generated by set $\mathcal{S}$.

For open sets $U \subset X$ and $V \subset Y$, we have

$\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are elements of the basis

for the product topology τ.

$\Rightarrow \pi_1^{-1}(U), \pi_2^{-1}(V)$ are open in τ.

Hence $\mathcal{S} \subset \tau$.

So arbitrary unions of finite intersections of elements of $\mathcal{S}$ are in τ.

Therefore, by Lemma 1, $\tau' \subset \tau$.

On the other hand, every basis element $U \times V$ for τ is of the form

$$U \times V = (U \times Y) \cap (X \times V) = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

(a finite intersection of elements of $\mathcal{S}$)

Thus $U \times V$ is in the topology τ' generated by $\mathcal{S}$.

That is, $\tau \subset \tau'$ and hence $\tau = \tau'$.

So the collection of all unions of finite intersections of $\mathcal{S}$ is τ.
Hence $\mathcal{S}$ is a subbasis for the product topology τ.

Definition 1.

Let X be a topological space with topology τ.

If Y is a subset of X, then the set

$$\tau_Y = \{Y \cap U \mid U \in \tau\}$$

is a topology on Y called the **subspace topology**.

With this topology, Y is called a **subspace** of X.

Lemma 1. If $\mathcal{B}$ is a basis for the topology of X then the set

$$\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. Let U be open in X.

Let $y \in U \cap Y$.

Since $\mathcal{B}$ is a basis for the topology of X, then there is a open set

$B \in \mathcal{B}$ such that $y \in B \subset U$.

Then $y \in B \cap Y \subset U \cap Y$.

By Lemma 2, $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Lemma 2. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof.

Let Y be a subspace of X.

Let U be open in Y.

Then by above Lemma $U = Y \cap V$ for some set V open in X.

Since Y and V are both open in X.

$\Rightarrow Y \cap V = U$ is open in X.

Lemma 3. If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof.

Let $U \times V$ be a basis element for the product topology on $X \times Y$.

Then $(U \times V) \cap (A \times B)$ is a basis element for the subspace topology on $A \times B$.

Now $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$.

Since $U \cap A$ and $V \cap B$ are open relative to A and B, respectively.

Then $(U \cap A) \times (V \cap B)$ is a basis element for the product topology on $A \times B$.

So the basis for the subspace topology on $A \times B$ is a subset of the basis for the product topology on $A \times B$.

Conversely,

A basis element for the product topology on $A \times B$ is of the form $(U \cap A) \times (V \cap B)$ where U and V are open in X and Y, respectively.

By the equality above, this is a basis element for

the subspace topology on $A \times B$.

So the basis for the product topology on $A \times B$ is a subset of the

basis for the subspace topology on $A \times B$.

Thus, the bases are the same and hence the topologies are the same.

Definition 2.

Given an ordered set X, a subset $Y \subset X$ is convex in X

if for each pair of points $a, b \in Y$ with $a < b$,

the entire interval (a, b) lies in Y.

Lemma 4. Let X be an ordered set in the order topology.

Let Y be a subset of X that is convex in X.

Then the order topology on Y is the same as the subspace

topology on Y.

Proof. By Theorem B, the set of all open rays form a subbasis

for the order topology on X.

Then the set $\mathcal{B}_S = \{(a, +\infty) \cap Y, Y \cap (-\infty, a) | a \in X\}$ is a subbasis

for the subspace topology on Y.

Since Y is convex then for $a \in Y$, we have

$(a, +\infty) \cap Y = \{a \in Y | x > a\}$ and $(-\infty, a) \cap Y = \{x \in Y | x < a\}$

and each of these is an open ray in Y.

If $a \notin Y$ then these two sets are either all of Y or are ϕ.

In all cases, each is open in the order topology and so

the order topology is a subset of the subspace topology.

Conversely, any open ray of Y equals the intersection of an open ray of X with Y and so is open in the subspace topology on Y.

Since the open rays of Y are a subbasis for the order topology on Y.

By Theorem B, this topology is a subset of the subspace topology.

Therefore, the subspace topology on Y is the same as the order topology on Y.

Definition 1. *Closed*

A subset A of a topological space X is closed if

set $X - A$ is open.

Example 1.

The subset $[a, b]$ of R is closed because its compliment

$R - [a, b] = (-\infty, a) \cup (b, +\infty)$ is open.

Theorem 1. Let X be a topological space.

Then the following conditions hold:

(1) ϕ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed

Proof of (1) follows: Since X and ϕ are open in X.

$\Rightarrow$ the compliments of ϕ and X are X and ϕ, respectively.

(i.e., $X - \phi = X$ and $X - X = \phi$)

Then by definition of closed, ϕ and X are closed in X.

(2): Given a collection of closed sets $\{A_\alpha\}$.

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha) \qquad \text{(by DeMorgans law)}$$

Since each A_α is closed. $\Rightarrow X - A_\alpha$ is open.

The right side of this equation is a union of open sets and so is open.

Therefore the left hand side is open.

By definition its compliment $\bigcap_{\alpha \in J} A_\alpha$ is closed.

(3): If A_i is closed for $i = 1, 2, ..., n$.

Consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i) \qquad \text{(by DeMorgans law)}$$

The set on the right side is a finite intersection of open sets

and is therefore open.

So the left hand side is open.

By definition its compliment $\bigcup_{i=1}^{n} A_i$ is closed.

Definition 2.

If Y is a subspace of X, we say that a set A is closed in Y

if $A \subset Y$ is closed in the subspace topology of Y

that is, $Y - A$ is open in the subspace topology of Y.

Theorem 2. Let Y be a subspace of X.

Then a set A is closed in Y if and only if it equals the intersection

of a closed set of X with Y.

Proof. Suppose $A = C \cap Y$ where C is closed in X.

Since C is closed in X, $X - C$ is open in X.

$\Rightarrow (X - C) \cap Y$ is open in Y.

(by the definition of the subspace topology).

But $(X - C) \cap Y = Y - A$ (the compliment of A in Y)

$\Rightarrow Y - A$ is open in Y. Hence A is closed in Y.

Conversely, suppose that A is closed in Y.

Then $Y - A$ is open in Y.

By definition of open in Y, there is an open set U in X such that

$Y - A = Y \cap U$.

$\Rightarrow X - U$ is closed in X.

But $A = Y \cap (X - U)$.

$\Rightarrow A$ is the intersection of Y and a closed set $X - U$ of X.

Theorem 3. Let Y be a subspace of X.

If A is closed in Y and Y is closed in X, then A is closed in X.

Proof

Given A is closed in Y.

By Theorem 2, $A = Y \cap C$ where C is closed in X.

$\Rightarrow Y \cap C$ closed in X. (since Y is closed in X, by Theorem 1)

$\Rightarrow A$ is closed in X.

Definition 3. Given a subset A of a topological space X.

The interior of A, denoted $Int(A)$, is the union of all

open subsets contained in A.

The closure of A, denoted $\bar{A}$ or $Cl(A)$, is the intersection of all

closed sets containing A.

Lemma A. Let A be a subset of topological space X.

Then A is open if and only if $A = Int(A)$.

A is closed if and only if $A = \bar{A}$.

Theorem 4. Let Y be a subspace of X.

Let $A \subset Y$ and denote the closure of A in X as $\bar{A}$.

Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof. Let B denote the closure of A in Y.

To prove $B = \bar{A} \cap Y$.

Since $\bar{A}$ is closed in X.

By Theorem 2, $\bar{A} \cap Y$ is closed in Y.

Given $A \subset Y$ and $A \subset \bar{A} \Rightarrow \bar{A} \cap Y$ contains A.

Since, by definition of closure, B equals the intersection of all

closed subsets of Y containing A.

$\Rightarrow B \subset \bar{A} \cap Y$.

On the other hand, B is closed in Y.

Hence by Theorem 2, $B = C \cap Y$ for some closed set C in X.

Then C is a closed set of X containing A. $(A \subset B \subset C)$

Now $\bar{A}$ is the intersection of all closed sets in X containing A.

$\Rightarrow \bar{A} \subset C \Rightarrow \bar{A} \cap Y \subset C \cap Y = B$.

$\Rightarrow \bar{A} \cap Y \subset B$

Thus, $B = \bar{A} \cap Y$.

Theorem 5. Let A be a subset of the topological space X.

(a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof (a). Consider the contrapositive.

i.e., $x \notin \bar{A}$ if and only if there is a neighborhood U of x that does not intersect A.

If $x \notin \bar{A}$ then the set $U = X - \bar{A}$ is a neighborhood of x

which does not intersect A, as claimed.

Conversely, if there is a neighborhood U of x which does not

intersect A.

Then $X - U$ is a closed set containing A.

By definition of the closure $\bar{A}$, the set $X - U$ must contain $\bar{A}$.

Since $x \in U$, then $x \notin \bar{A}$.

Proof (b). Suppose $x \in \bar{A}$.

Then by part (a), every neighborhood of x intersects A.

Then every basis element B containing x intersects A.

(since each B is open).

Conversely, if every basis element containing x intersects A.

Then every neighborhood U of x, $\Rightarrow$ U contains a basis element

that contains x.

i.e., every neighborhood U of x intersects A.

i.e., $x \in \bar{A}$.

Definition 4.

If A is a subset of topological space X and if $x \in X$

then x is a limit point (or cluster point or point of accumulation) of A

if every neighborhood of x intersects A in some point other than

x itself.

Theorem 6. Let A be a subset of the topological space X.

Let A' be the set of all limit points of A.

Then $\bar{A} = A \cup A'$.

Proof. If $x \in A'$ then every neighborhood of x intersects A in a

point different from x.

Therefore, by Theorem $17.5(a), x$ belongs to $\bar{A}$.

Hence $A' \subset \bar{A}$.

Since $A \subset \bar{A}$, we have $A \cup A' \subset \bar{A}$.

Let $x \in \bar{A}$.

If $x \in A$, then $x \in A \cup A'$. $(\Rightarrow \bar{A} \subset A \cup A'.)$

If $x \notin A$ then, since $x \in \bar{A}$, every neighborhood U of x intersects A.

Because $x \in \bar{A}$ then U must intersect A in a point different from x.

Then $x \in A'$.

so that $x \in A \cup A'$.

Therefore, $\bar{A} \subset A \cup A'$.

Hence $\bar{A} = A \cup A'$, as claimed.

Corollary 7. A subset of a topological space is closed if and only if

it contains all its limit points. (i.e., $A = \bar{A}$)

Proof.

By Lemma 17.A, the set A is closed if and only if $A = \bar{A}$.

By Theorem 17.6, $\bar{A} = A \cup A'$.

$\Rightarrow A = \bar{A}$ if and only if $A' \subset A$.

i.e., A is closed in X, if and only if it contains all its limit points.

Definition 5.

A topological space X is a Hausdorff space if for each pair of distinct points $x_1, x_2 \in X$, there exist neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \phi$.

Theorem 8. Every finite point set in a Hausdorff space X is closed.

Proof. Consider the set $\{x_0\}$.

Consider $x \in X$ where $x \neq x_0$.

Since X is a Hausdorff space, there are disjoint neighborhoods

U of x and V of x_0.

$\Rightarrow U$ does not intersect $\{x_0\}$.

By Theorem 5(a), x is not in the closure of set $\{x_0\}$.

Since $x \neq x_0$ is an arbitrary element of X, the only points of

closure of $\{x_0\}$ is x_0 itself.

By Corollary 7, $\{x_0\}$ is a closed set.

Now consider a finite point set, say $\{x_0, x_1, ..., x_n\}$.

Write the set as $\{x_0\} \cup \{x_1\} \cup \ldots \cup \{x_n\}$.

Observe that each $\{x_i\}$ is closed in X.

Apply Theorem 17.1 part (3), $\{x_0, x_1, ..., x_n\}$ is closed.

Thus, every finite point set in a Hausdorff space X is closed.

Theorem 9. Let X be a space satisfying the T_1 Axiom.

Let A be a subset of X.

Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. Suppose every neighborhood of x intersects A in infinitely many points.

Then every neighborhood of x intersects set A at a point other than x.

By definition, x is a limit point of A.

Conversely, suppose that x is a limit point of A.

Assume some neighborhood U of x intersects A in

only finitely many points.

Then U also intersects $A - \{x\}$ in finitely many points.

Say $\{x_1, x_2, ..., x_m\} = U \cap (A - \{x\})$.

Since X satisfy T_1 Axiom,$\Rightarrow \{x_1, x_2, ..., x_m\}$ is closed.

Therefore, set $X - \{x_1, x_2, ..., x_m\}$ is open in X.

Then $U \cap (X - \{x_1, x_2, ..., x_m\})$ is a neighborhood of x that

does not intersect the set $A - \{x\}$.

But this CONTRADICTS the hypothesis that x is a limit point of A.

So the assumption that U intersects A in finitely many points is false.

That is, any neighborhood of x must intersect A in infinitely

many points.

Theorem 10. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Proof.

Let $\{x_n\}$ be a sequence of points of X that converges to x.

Let $y \neq x$.

Let U and V be disjoint neighborhoods of x and y, respectively.

Since U is a neighborhood of x, then there is $N_1 \in \mathbb{N}$ such that $x_n \in U$ for all $n \in N_1$.

So there is no $N_2 \in \mathbb{N}$ such that for $n \in N_2$ we have $x_n \in V$

Since for $n \in N_1, \; x_n \in U$ and $U \cap V = \phi$.

That is, x_n does not converge to $y \neq x$.

Thus, x_n converges to at most one point x in X. $(x_n \to x)$

Theorem 11.

(a). Every simply ordered set is a Hausdorff space in the order topology.

(b). The product of two Hausdorff spaces is a Hausdorff space.

(c). A subspace of a Hausdorff space is a Hausdorff space.

Proof (a). Suppose τ is an order topology on a given set X.

Let x_1, x_2 be distinct points in X where $x_1 < x_2$.

If x_2 is not the immediate successor of x_1, there is some $c \in (x_1, x_2)$.

If x_1 and x_2 are not the smallest or largest elements of X, respectively.

Then there is some $a < x_1$ and $b > x_2$.

It follows that (a, c) and (c, b) are neighborhoods of x_1 and x_2

that are disjoint.

On the other hand, if (x_1, x_2) is empty, then (a, x_1) and (x_2, b)

are the appropriate disjoint neighborhoods.

If x_1 is the smallest element of X.

By the same argument as above.

Let us consider the neighborhood of x_1 be $[x_1, c)$ or $[x_1, x_2)$,

as appropriate.

Similarly, if x_2 is the largest element of X.

Then the neighborhood of x_2 be $(c, x_2]$ or $(x_1, x_2]$, as appropriate.

Hence, every order topology is Hausdorff.

Proof (c). Let X be a Hausdorff space and Y a subset of X.

Given any distinct x_0, x_1 in $Y \subset X$.

Then there are neighborhoods U of x_0 and V of x_1 in X

that are disjoint.

By definition, $U' = U \cap Y$ and $V' = V \cap Y$ are open in Y.

Now $U' \cap V' = (U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \phi$.

Hence, U' and V' are disjoint neighborhoods of x_0 and x_1 in Y.

Hence, the subspace Y is Hausdorff.

Proof (b). Suppose X and Y are Hausdorff spaces.

Given distinct (x_0, y_0), (x_1, y_1) of $X \times Y$, if $x_0 \neq x_1$ and $y_0 \neq y_1$.

Then there are neighborhoods A_0 of x_0 and A_1 of x_1 and

neighborhoods B_0 of y_0 and B_1 of y_1 that are disjoint.

Consider, $(A_0 \times B_0) \cap (A_1 \times B_1) = (A_0 \cap A_1) \times (B_0 \cap B_1) = \phi$.

Therefore $A_0 \times B_0$ and $A_1 \times B_1$ are disjoint neighborhoods

of (x_0, y_0) and (x_1, y_1).

On the other hand, if $x_0 = x_1$ (in which case $y_0 \neq y_1$).

Let A be any neighborhood of x_0 and B_0 and B_1 be as above.

Consider, $(A \times B_0) \cap (A \times B_1) = (A \cap A) \times (B_0 \cap B_1) = A \cap \phi = \phi$.

Therefore $A \times B_0$ and $A \times B_1$ are disjoint neighborhoods

of (x_0, y_0) and (x_0, y_1).

Similarly, there exist disjoint neighborhoods for (x_0, y_0) and

(x_1, y_0) where $x_0 \neq x_1$.

Thus, the product of two Hausdorff spaces is Hausdorff.