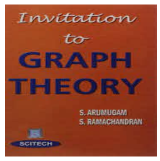


I. Trees

Definition 1. (Acyclic graph).
An *acyclic graph* (or a *forest*) is one that contains no cycles.

Definition 2. (Tree).
A *tree* is a connected acyclic graph.

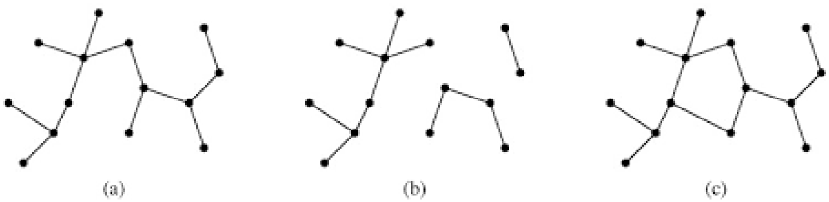


Figure 1: (a) Tree (b) Forest (c) Graph

Remark.
(i) Each component of an acyclic graph is a tree.
(ii) An acyclic graph is a simple graph. Hence, every tree is a simple graph.
(iii) A subgraph of a tree is an acyclic graph.

Remark.
If $e \in E(G)$, then $\omega(G-e) = \omega(G)$ or $\omega(G-e) = \omega(G)+1$.

Definition 3. (spanning tree)
A spanning subgraph of a graph, which is also a tree, is called a *spanning tree* of the graph.

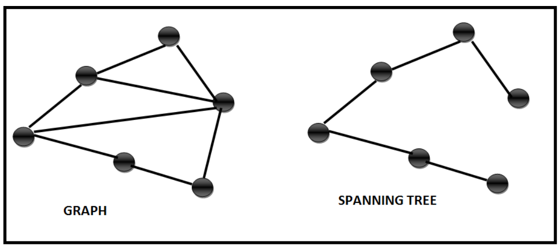
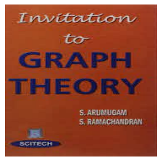


Figure 2: Spanning tree

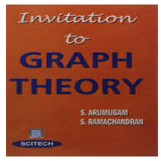


Theorem 1. Let $G = (p, q)$ graph. The following statments are equivalent.

1. G is a tree.
2. Every two vertices of G are connected by a unique path.
3. G is conneced and $p = q + 1$.
4. G is acyclic and $p = q + 1$.

Proof. (1)⇒(2):
Assume G be a tree.
By definition, G is connected
Therefore any two vertices of G are connected by a path.
To prove : Any two vertices of G are connected by a unique path.
Proof by contradiction.
Assume that there are two distinct (u, v)-paths P_1 and P_2 in G.
Path P_1 is : uv
Path P_2 is : uP_2v
Clearly, the graph $(P_1 \cup P_2)$ is connected.
But then $(P_1 \cup P_2) = uP_2vu$ is a cycle in G, a contradiction to G is acyclic.
Thus, every two vertices of G are connected by a unique path.

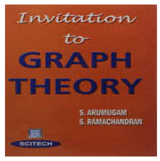
(2)⇒(3):
Assume every two vertices of G are connected by a unique path.
To prove : G is conneced and $p = q + 1$.
Since every two vertices of G are connected by a unique path, ⇒ G is connected.
Now we prove that $q = p - 1$.
Proof by induction on p.
If $p = 1$, $G \cong K_1$ and therefore $q = 0$. Hence $p - 1 = 1 - 1 = 0 = q$.
Suppose that the theorem is true for all graphs G on fewer than p vertices.
Let G be a connected graph on $p \geq 2$ vertices.
Let $e = uv \in E(G)$.
Consider $G - e$.
As G is conneced and any two vertices of G are connected by a unique path.
⇒ uev is the unique (u, v)-path in G
⇒ $G - e$ contains no (u, v)-path.
⇒ $G - e$ is disconnected.
G is connected and $G - e$ is disconnected implies that
$\omega(G - e) = \omega(G) + 1 = 1 + 1 = 2$.
Let G_1 and G_2 be the two components of $G - e$.
Both G_1 and G_2 are components ⇒ both G_1 and G_2 are connected.
Moreover, $p(G_1) < p(G)$ and $p(G_2) < p(G)$.
Therefore, by the induction hypothesis, $q(G_1) = p(G_1) - 1$ and $q(G_2) = p(G_2) - 1$.
But $q(G) = q(G_1) + q(G_2) + 1$ and $p(G) = p(G_1) + p(G_2)$.



Therefore, $q(G) = q(G_1) + q(G_2) + 1 = (p(G_1) - 1) + (p(G_2) - 1) + 1$
$= p(G_1) + p(G_2) - 1 = p(G) - 1.$
Hence $q(G) = p(G) - 1$.

(3)$\Rightarrow$(4):
Assume : G is conneced and $p = q + 1$.
To prove : G is acyclic.
i.e., to prove there is no cycle in G.
Proof by cotradiction..
Suppose G contains a cycle of length $n \geq 3$.
$\Rightarrow p = p - n + n$ where n vertices belongs to C_n and $p - n$ vertices not in C_n.
Fix a vertex u in the cycle C_n.
Let v be the vertex not in C_n. (there are $p - n$ vertices are remaining in G)
Since G is connected, there exits a shortest $(u.v)$-path in G.
Let e be the edge on this shortest path incident with v.
Clearly, we obtained $p - n$ distinct edges in G.
$p = q + 1 \Rightarrow q = p - 1$.
Now, $p - 1 = q \geq n + (p - n) = p \Rightarrow p - 1 \geq p$, a cotradiction.
Hence G is acyclic.

(4)$\Rightarrow$(1):
Assume G is acyclic.
To prove G is a tree.
i.e., to prove G is connected.
Proof by cotradiction.
Suppose G is not conneced.
Then G has more than one component.
Let $G_1, G_2, \ldots, G_k, k \geq 2$ be the components of G.
Each component is connected and G is acyclic, $\Rightarrow$ Each $G_i, i \geq 2$ is connected and acyclic.
$\Rightarrow$ Each $G_i = (p_i, q_i), i \geq 2$ is a tree.
$\Rightarrow p_i = q_i + 1$ for all i, $1 \leq i\, k$.
But, $p = p_1 + p_2 + \ldots + p_k$
Therefore, $p = p_1 + p_2 + \ldots + p_k = (q_1 + 1) + (q_2 + 1) + \ldots + (q_k + 1)$
i.e., $p = q_1 + q_2 + \ldots + q_k + k$, a cotradiction. (since $p = q + 1$)
Thus, G is connected.
Hence G is a tree.



Theorem 2. Every connected graph contains a spanning tree.

Proof. Let G be a connected graph.
Let $\mathcal{C}$ be the collection of all connected spanning subgraphs of G.
Clearely, $\mathcal{C} \neq \phi$ (since $G \in \mathcal{C}$).
Let $T \in \mathcal{C}$ be the connected spanning subgraph with least number of edges.
To prove : T be the spanning tree in G.
Suppose T contains no cycle.
$\Rightarrow$ T be a spanning tree of G, then the theorem is complete.
Otherwise, T contain a cycle of G
Then $T - e$ is connected.
$\Rightarrow$ $T - e \in \mathcal{C}$, a contradicts the choice of T.
Hence T has no cycles.
Thus, T be the spanning tree in G.

Theorem 3. Every nontrivial tree has at least two vertices of degree one.

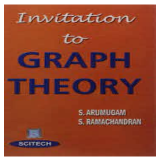
Proof. Let T be a nontrivial tree.
Then $d_T(v) \geq 1$ for all $v \in V(T)$.
(If $d_T(v) = 0$ for some vertex v, then T is the trivial tree K_1, a contradiction.)
Since T is a tree, $\Rightarrow$ $m(T) = n(T) - 1$.
By Euler's theorem, $\sum_{v \in V(T)} d_T(v) = 2m(T)$
Hence $\sum_{v \in V(T)} d_T(v) = 2m(T) = 2(n(T) - 1) = 2n(T) - 2$.
Proof by contradiction.
Suppose $d_T(v) \geq 2$ for all $v \in V(T)$, then
$2n(T) - 2 = \sum_{v \in V(T)} d_T(v) \geq (2 + 2 + \ldots + 2)(n(T)\ times) = 2n(T)$.
$\Rightarrow$ $2n(T) - 2 \geq 2n(T)$, a contradiction.
So there is a vertex, say, x such that $d_T(x) = 1$.
If $d_T(v) \geq 2$ for all $v \neq x$ and $v \in V(T)$, then

$$\begin{aligned} 2n(T) - 2 &= \sum_{v \in V(T)} d_T(v) = 1 + \sum_{v \in V(T),\, v \neq x} d_T(v) \\ &\geq 1 + (2 + 2 + \ldots + 2)(n(T) - 1\ times) \\ &= 1 + 2(n(T) - 1) = 1 + 2n(T) - 2 = 2n(T) - 1. \end{aligned}$$

$\Rightarrow$ $2n(T) - 2 \geq 2n(T) - 1$, a contradiction.
So there is a vertex, say, y, $y \neq x$ such that $d_T(y) = 1$.
Hence T has at least two vertices of degree one.

Theorem 4. If u and v are nonadjacent vertices of a tree T.
Then $T + uv$ contains a unique cycle.

Proof. If P is the unique $u - v$ path in T
Then $P + uv$ is a cycle in $T + uv$.
As a path P is unique in T, $P + uv$ is a unique cycle in $T + uv$.



II. Distances in Graphs

Definition 4. (subtree)
A connected subgraph of a tree T is a *subtree* of T.

Definition 5. (Distance).
If vertices u and v are connected in G, the *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G.
If there is no path connecting u and v in G, define $d_G(u, v)$ to be infinite.

Definition 6. (eccentricity, radius, center)
Let G be a connected graph.
(i). If v is a vertex of G, its *eccentricity* $e_G(v)$ is defined by
$e_G(v) = \max\{d_G(v, u) : u \in V(G)\}$.
(ii). The *radius* of G, $r(G)$, is the minimum eccentricity of G, that is
$r(G) = \min\{e_G(v) : v \in V(G)\}$.
(iii). The *diameter* of G, $diam(G)$, is the maximum eccentricity of G, that is
$diam(G) = \max\{e_G(v) : v \in V(G)\}$.
(iv). A vertex v of G is called a *central vertex* if $e_G(v) = r(G)$.
(v). The set of all central vertices of G is called the *center* of G.

Remark. It is obvious from the definition that $r(G) \leq diam(G)$.

Examples.
(i). For the complete graph K_n,
$r(K_n) = diam(K_n) = 1$, since $d_{K_n}(v, u) = 1$ $(u \neq v)$.
(ii). For the complete bipartite graph $K_{m,n}$ with $\min\{m, n\} \geq 2$,
$r(K_{m,n}) = diam(K_{m,n}) = 2$.
(iii). For the Petersen graph P, $r(P) = diam(P) = 2$.

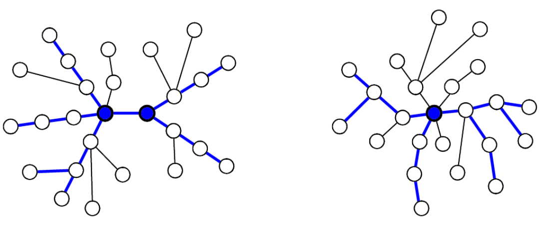
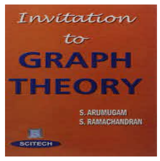


Figure 3: Centre K_1 or K_2



Theorem 5. (Jordan). Every tree has a center consisting of either a single vertex or two adjacent vertices.

Proof. The result if obvious for the trees K_1 and K_2.
The vertices of K_1 and K_2 are central vertices.
Now let T be a tree with $n(T) \geq 3$.
Then T has at least two pendant vertices.
Clearly, the pendant vertices of T cannot be central vertices.
Delete all pendant vertices from T.
This results, a subtree T' of T.
As any maximum distance path in T form any vertex of T' ends at a pendant vertex of T.
The eccentricity of each vertex of T' is one less than the eccentricity of the same vertex in T.
Hence, the vertices of minimum eccentricity of T' are the same as those of T.
In other words, T and T' have the same center.
Now if T'' is the tree obtained from T' by deleting all the pendant vertices of T', then T'' and T' have the same center.
Hence the centers of T'' and T are the same.
Since T is finite, repeat the process of deleting the pendant vertices in the successive subtrees of T until there results a K_1 or K_2.
Hence, the center of T is either a single vertex or a pair of adjacent vertices.