BINOMIAL SERIES

$$(x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \cdots + nc_r x^{n-r} a^r + \cdots$$

$$(1+x)^n = 1 + nc_1 x + nc_2 x^2 + nc_3 x^3 + \cdots nc_r x^r + \cdots x^n$$

Binomial Theorem for a rational Index

$$(1+x)^n = 1 + \frac{n}{\lfloor 1} x + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \cdots$$
$$+ \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} x^r + \cdots$$

Note:

It can be shown that the series is also convergent to the sum $(1+x)^n$ in the following cases.

(i) if $x = 1$ and $n > -1$

if $x = 1$ and $n > 0$

2) The general term of the series is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}$$

RHS is denoted by $\frac{n_r}{\lfloor r}$.

The reader should be familiar with the following expansions which can be arrived at from the general binomial series given in (1)

$$(1-x)^{-1} = 1 + x + x^2 + \cdots + x^n + \cdots \infty$$

$$(1+x)^{-1} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots \infty$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots (n+1) x^n + \cdots \infty$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - \cdots (-1)^n (n+1) x^n + \cdots \infty$$

$$(1-x)^{-3} = \frac{1}{2}\left[1\cdot2 + 2\cdot3x + 3\cdot4x^2 + \cdots + (n+1)(n+2)x^n + \cdots \infty\right]$$

$$(1+x)^{-3} = \frac{1}{2}\left[1\cdot2 - 2\cdot3x + 3\cdot4x^2 - \cdots + (-1)^n(n+1)(n+2)x^n + \cdots \infty\right]$$

$$(1-x)^{-4} = \frac{1}{6}\left[1\cdot2\cdot3 + 2\cdot3\cdot4x + 3\cdot4\cdot5x^2 + \cdots + (n+1)(n+2)(n+3)x^n \cdots \infty\right]$$

$$(1+x)^{-4} = \frac{1}{6}\left[1\cdot2\cdot3 - 2\cdot3\cdot4x + 3\cdot4\cdot5x^2 - \cdots + (-1)^n(n+1)(n+2)(n+3)x^n \cdots \infty\right]$$

IN general

$$(1-x)^{-p/q} = 1 + \frac{\left(-\frac{p}{q}\right)}{\underline{|1}}(-x) + \frac{\left(-\frac{p}{q}\right)\left(-\frac{p}{q}-1\right)}{\underline{|2}}(-x)^2 + \frac{\left(-\frac{p}{q}\right)\left(-\frac{p}{q}-1\right)\left(-\frac{p}{q}-2\right)}{\underline{|3}}(-x)^3 + \cdots \infty$$

$$\therefore (1-x)^{-p/q} = 1 + \frac{p}{\underline{|1}}\left(\frac{x}{q}\right) + \frac{p(p+q)}{\underline{|2}}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{\underline{|3}}\left(\frac{x}{q}\right)^3 + \cdots \infty$$

Again,

$$(1+x)^{-p/q} = 1 - \frac{p}{\underline{|1}}\left(\frac{x}{q}\right) + \frac{p(p+q)}{\underline{|2}}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{\underline{|3}}\left(\frac{x}{q}\right)^3 + \cdots \infty$$

It will also be useful to remember the following results

$$(1-x)^{-n} = 1 + \frac{n}{\underline{|1}}x + \frac{n(n+1)}{\underline{|2}}x^2 + \frac{n(n+1)(n+2)}{\underline{|3}}x^3 + \cdots \infty$$

$$(1+x)^{-n} = 1 - \frac{n}{\underline{|1}}x + \frac{n(n+1)}{\underline{|2}}x^2 - \frac{n(n+1)(n+2)}{\underline{|3}}x^3 + \cdots \infty$$

$$(1+x)^{n} = 1 + \frac{n}{\underline{|1}}x + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots$$

$$(1-x)^{n} = 1 + \frac{n}{\underline{|1}}x + \frac{n(n-1)}{\underline{|2}}x^2 - \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots$$

Ex: 1 Find the Sum to infinity of the Series

$$1+\frac{2}{6}+\frac{2\cdot 5}{6\cdot 12}+\frac{2\cdot 5\cdot 8}{6\cdot 12\cdot 18}+\cdots\infty$$

Sol: -

Let $S=1+\frac{2}{6}+\frac{2\cdot 5}{6\cdot 12}+\frac{2\cdot 5\cdot 8}{6\cdot 12\cdot 18}+\cdots\infty$

$$S=1+\frac{2}{1!}\left(\frac{1}{6}\right)+\frac{2\cdot 5}{2!}\left(\frac{1}{6}\right)^2+\frac{2\cdot 5\cdot 8}{3!}\left(\frac{1}{6}\right)^3+\cdots\infty \rightarrow ①$$

We know that

$$(1-x)^{-p/q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\cdots\infty$$

Comparing ① and ②

$\boxed{p=2}$

$p+q=5$

$2+q=5$

$\boxed{q=3}$

$\frac{x}{q}=\frac{1}{6}$

$\frac{x}{3}=\frac{1}{6}$

$x=\frac{3}{6} \rightarrow \boxed{x=\frac{1}{2}}$

$$S=(1-x)^{-p/q}$$

$$=(1-1/2)^{-2/3}$$

$$=(1/2)^{-2/3}$$

$$=(2^{-1})^{-2/3}$$

$$=(2)^{2/3}$$

$$\boxed{S=(4)^{1/3}}$$

EX2 Sum to infinity the Series $\frac{2\cdot 4}{3\cdot 6}+\frac{2\cdot 4\cdot 6}{3\cdot 6\cdot 9}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 6\cdot 9\cdot 12}+\cdots\infty$

Sol:

Let $S=\frac{2\cdot 4}{3\cdot 6}+\frac{2\cdot 4\cdot 6}{3\cdot 6\cdot 9}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 6\cdot 9\cdot 12}+\cdots\infty$

$$S = \frac{2\cdot4}{\lfloor 2}\left(\frac{1}{3^2}\right) + \frac{2\cdot4\cdot6}{\lfloor 3}\left(\frac{1}{3^3}\right) + \frac{2\cdot4\cdot6\cdot8}{\lfloor 4}\left(\frac{1}{3^4}\right) + \cdots \infty$$

Add $1 + \frac{2}{\lfloor 1}\left(\frac{1}{3}\right)$ to both sides

$$S + 1 + \frac{2}{\lfloor 1}\left(\frac{1}{3}\right) = 1 + \frac{2}{\lfloor 1}\left(\frac{1}{3}\right) + \frac{2\cdot4}{\lfloor 2}\left(\frac{1}{3^2}\right) + \frac{2\cdot4\cdot6}{\lfloor 3}\left(\frac{1}{3^3}\right) + \frac{2\cdot4\cdot6\cdot8}{\lfloor 4}\left(\frac{1}{3^4}\right) + \cdots \infty \quad (1)$$

We know that

$$(1-x)^{-p/q} = 1 + \frac{p}{\lfloor 1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{\lfloor 2}\left(\frac{x}{q}\right)^2 + \frac{p(p+2q)}{\lfloor 3}\left(\frac{x}{q}\right)^3 + \cdots \infty \quad (2)$$

Comparing (1) and (2)

$\boxed{p=2}$

$p+q=4$

$2+q=4$

$\Rightarrow \boxed{q=2}$

$\frac{x}{q} = \frac{1}{3}$

$\frac{x}{2} = \frac{1}{3}$

$\boxed{x = \frac{2}{3}}$

$$S + 1 + \frac{2}{3} = (1-x)^{-p/q}$$

$$= (1 - 2/3)^{-2/2}$$

$$= (1 - 2/3)^{-1}$$

$$= \left(\frac{3-2}{3}\right)^{-1}$$

$$= \left(\frac{1}{3}\right)^{-1}$$

$$S + \frac{5}{3} = (3^{-1})^{-1}$$

$$\boxed{S = 3 - \frac{5}{3}}$$

$$\Rightarrow \boxed{S = \frac{4}{3}}$$

(3) Ex 3

Sum to ∞ : $\frac{4}{2\cdot4} + \frac{4\cdot5}{2\cdot4\cdot6} + \frac{4\cdot5\cdot6}{2\cdot4\cdot6\cdot8} + \cdots\cdots$

Let $S = \frac{4}{2\cdot4} + \frac{4\cdot5}{2\cdot4\cdot6} + \frac{4\cdot5\cdot6}{2\cdot4\cdot6\cdot8} + \cdots\cdots\infty$

$3\times \Rightarrow \quad 3S = \frac{3\cdot4}{2\cdot4} + \frac{3\cdot4\cdot5}{2\cdot4\cdot6} + \frac{3\cdot4\cdot5\cdot6}{2\cdot4\cdot6\cdot8} + \cdots\cdots\infty$

$3S = \frac{3\cdot4}{\lfloor 2}\left(\frac{1}{2^2}\right) + \frac{3\cdot4\cdot5}{\lfloor 3}\left(\frac{1}{2^3}\right) + \frac{3\cdot4\cdot5\cdot6}{\lfloor 4}\left(\frac{1}{2^4}\right) \cdots$

Add $1 + \frac{3}{\lfloor 1}\left(\frac{1}{2}\right)$ to both sides

$$3S + 1 + \frac{3}{\lfloor 1}\left(\frac{1}{2}\right) = 1 + \frac{3}{\lfloor 1}\left(\frac{1}{2}\right) + \frac{3\cdot4}{\lfloor 2}\left(\frac{1}{2^2}\right) + \frac{3\cdot4\cdot5}{\lfloor 3}\left(\frac{1}{2^3}\right) + \frac{3\cdot4\cdot5\cdot6}{\lfloor 4}\left(\frac{1}{2^4}\right) + \cdots\infty \quad (1)$$

WKT

$$(1-x)^{-p/q} = 1 + \frac{p}{\lfloor 1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{\lfloor 2}\left(\frac{x}{q}\right)^2 + \frac{p(p+2q)}{\lfloor 3}\left(\frac{x}{q}\right)^3 + \cdots\infty \quad (2)$$

Comparing (1) and (2)

$\boxed{p = 3}$

$p + q = 4$

$3 + q = 4$

$\boxed{q = 1}$

$\frac{x}{q} = \frac{1}{2}$

$\boxed{x = \frac{1}{2}}$

$3S + 1 + \frac{3}{\lfloor 1}\left(\frac{1}{2}\right) = (1-x)^{-p/q}$

$3S + 1 + \frac{3}{2} = \left(1 - \frac{1}{2}\right)^{-3/1}$

$3S + \frac{5}{2} = \left(1 - \frac{1}{2}\right)^{-3}$

$3S = (1)^{+3} - \frac{5}{2} \Rightarrow \frac{1}{1^{-\infty}}$

$3S = \frac{8-5}{2} \qquad \therefore S = \frac{1}{6}$

Sum to infinity the series

$$\frac{1\cdot 3}{2\cdot 4\cdot 6\cdot 8} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \cdots \infty$$

Sol:

$$\text{Let } S = \frac{1\cdot 3}{2\cdot 4\cdot 6\cdot 8} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \cdots \infty$$

$$\times(-3)(-1) \Rightarrow 3S = \frac{(-3)(-1)\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8} + \frac{(-3)(-1)\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10} + \cdots \infty$$

$$3S = \frac{(-3)(-1)\cdot 1\cdot 3}{\lfloor 4}\left(\frac{1}{2^4}\right) + \frac{(-3)(-1)\cdot 1\cdot 3\cdot 5}{\lfloor 5}\left(\frac{1}{2^5}\right) + \cdots \infty$$

Add $1 + \frac{(-3)}{\lfloor 1}\frac{1}{2} + \frac{(-3)(-1)}{\lfloor 2}\frac{1}{2^2} + \frac{(-3)(-1)\cdot 1}{\lfloor 3}\frac{1}{2^3}$ both sides

$$3S + 1 + \frac{(-3)}{1}\cdot\frac{1}{2} + \frac{(-3)(-1)}{\lfloor 2}\left(\frac{1}{2^2}\right) + \frac{(-3)(-1)\cdot 1}{\lfloor 3}\left(\frac{1}{2^3}\right)$$

$$= 1 + \frac{(-3)}{1}\left(\frac{1}{2}\right) + \frac{(-3)(-1)}{\lfloor 2}\left(\frac{1}{2^2}\right) + \frac{(-3)(-1)}{\lfloor 3}\left(\frac{1}{2^2}\right) + \frac{(-3)(-1)\cdot 1}{\lfloor 3}\frac{1}{2^3} + \cdots \infty$$

WKT

$$(1-x)^{-p/q} = 1 + \frac{p}{\lfloor 1}\left(\frac{x}{q}\right) + \frac{p+q}{\lfloor 2} + \left(\frac{x}{q}\right)^2 + \frac{p+2q}{\lfloor 3}\left(\frac{x}{q}\right)^3 + \cdots \infty$$

Comparing ① and ②

$p = -3$

$p + q = -1$

$-3 + q - 1$

$q = -1 + 3$

$\boxed{q = 2}$

$\frac{x}{q} = \frac{1}{2}$

$\frac{x}{2} = \frac{1}{2}$

$\boxed{x = 1}$

$$3S + 1 - \frac{3}{2} + \frac{3}{8} + \frac{1}{16} = (1-x)^{-p/q} = (1-1)^{3/2}$$

$$3S = -1 + \frac{3}{2} - \frac{3}{8} - \frac{1}{16} + 0$$

$$S = \frac{1}{48}$$

Ex 5 Sum to infinity $\frac{11 \cdot 14}{10 \cdot 15 \cdot 20} + \frac{11 \cdot 14 \cdot 17}{10 \cdot 15 \cdot 20 \cdot 25} + \cdots \infty$

$$\text{LHS} = \frac{11 \cdot 14}{10 \cdot 15 \cdot 20} + \frac{11 \cdot 14 \cdot 17}{10 \cdot 15 \cdot 20 \cdot 25} \cdots\cdots P$$

$$\times \text{ by } 8 \Rightarrow \quad 8S = \frac{8 \cdot 11 \cdot 14}{10 \cdot 15 \cdot 20} + \frac{11 \cdot 14 \cdot 17}{10 \cdot 15 \cdot 20 \cdot 25} + \cdots \infty$$

$$8S = \frac{8 \cdot 11 \cdot 14}{\underline{|4}} \frac{1}{5^3} + \frac{11 \cdot 14 \cdot 17}{\cancel{10 \cdot 15 \cdot 20 \cdot 25}\ \underline{|5}} \frac{1}{5^4} + \cdots \infty$$

$$\frac{5 \cdot 8S}{5} = \frac{5 \cdot 8 \cdot 11 \cdot 14}{\underline{|4}} \frac{1}{5^4} + \frac{5 \cdot 8 \cdot 11 \cdot 14 \cdot 17}{\underline{|5}} \frac{1}{5^5} + \cdots \infty$$

Add both side $1 + \frac{5}{\underline{|1}} \frac{1}{5} + \frac{5 \cdot 8}{\underline{|2}} \frac{1}{5^2} + \frac{5 \cdot 8 \cdot 11}{\underline{|3}} \frac{1}{5^3}$

$$8S + 1 + \frac{5}{\underline{|1}}\left(\frac{1}{5}\right) + \frac{5 \cdot 8}{\underline{|2}}\left(\frac{1}{5^2}\right) + \frac{5 \cdot 8 \cdot 11}{\underline{|3}}\left(\frac{1}{5^3}\right)$$

$$= 1 + \frac{5}{\underline{|1}}\left(\frac{1}{5}\right) + \frac{5 \cdot 8}{\underline{|2}}\left(\frac{1}{5^2}\right) + \frac{5 \cdot 8 \cdot 11}{\underline{|3}}\left(\frac{1}{5^3}\right) + \frac{5 \cdot 8 \cdot 11 \cdot 14}{\underline{|4}}\left(\frac{1}{5^4}\right) \cdots \infty$$

WKT

$$(1-x)^{-p/q} = 1 + \frac{p}{\underline{|1}}\left(\frac{x}{q}\right) + \frac{p+q}{\underline{|2}}\left(\frac{x}{q}\right)^2 + \frac{p+2q}{\underline{|3}}\left(\frac{x}{q}\right)^3 + \cdots \infty$$

Comparing ③ and ⑤

$p=5, \quad q=3, \quad \frac{x}{q}=\frac{1}{5}$

$\therefore x=\frac{3}{5}$

$8S+2+\frac{4}{5}+\frac{44}{75}=(1-x)^{-p/q}$

$=(1-3/5)^{-5/3}$

$=\left(\frac{2}{5}\right)^{-5/3}$

$8S+\frac{254}{75}=\left(\frac{5}{2}\right)^{5/3}$

$8S=\left(\frac{5}{2}\right)^{5/3}-\frac{254}{75}$

$S=\frac{1}{8}\left[\frac{5}{2}\left(\frac{5}{2}\right)^{2/3}-\frac{254}{75}\right]$

$S=\frac{1}{8}\left[\frac{5}{2}\left(\frac{25}{4}\right)^{1/3}-\frac{254}{75}\right]$

Ex:6 Sum to infinity the series $\frac{7}{72}+\frac{7\cdot 28}{72\cdot 96}+\frac{7\cdot 28\cdot 49}{72\cdot 96\cdot 120}+\dots\infty$

Sol:

Let $S=\frac{7}{72}+\frac{7\cdot 28}{72\cdot 96}+\frac{7\cdot 28\cdot 49}{72\cdot 96\cdot 120}+\dots\infty$

$=\frac{7}{3}\cdot\frac{1}{24}+\frac{7\cdot 28}{3\cdot 4}\cdot\frac{1}{24^2}+\frac{7\cdot 28\cdot 49}{3\cdot 4\cdot 5}\cdot\frac{1}{24^3}+\dots\infty$

$\times\frac{1}{2}\quad \frac{S}{2}=\frac{1\cdot 7}{\lfloor 3}\frac{1}{24}+\frac{1\cdot 7\cdot 28}{\lfloor 4}\frac{1}{24^2}+\dots$

$\frac{S}{2\cdot 24^2}=\frac{7}{\lfloor 3}\cdot\frac{1}{24^3}+\frac{7\cdot 28}{\lfloor 4}+\frac{1}{24^2}+\dots$

$$\frac{(-35)(-14)}{2\cdot 24^3}\cdot S = \frac{(-35)(-14)\,7}{\underline{|3}}\cdot\frac{1}{24^3} + \frac{(-35)(-14)(\ldots)}{\underline{|4}}$$

$$\frac{(-35)(-14)}{1\cdot 2\cdot 24^2}\cdot S + 1 + \frac{(-35)}{\underline{|1}}\cdot\frac{1}{24} + \frac{(-35)(-14)}{\underline{|2}}\cdot\frac{1}{24^2}$$

$$= 1 + \frac{(-35)}{\underline{|1}}\cdot\frac{1}{24} + \frac{(-35)(-14)}{\underline{|2}}\frac{1}{24^2} + \frac{(-35)(-14)\,7}{\underline{|3}}\cdot\frac{1}{24^3}$$

$p = -35,\ q = 21;\ \frac{x}{q} = \frac{1}{24};\ \therefore x = \frac{21}{24} = \frac{7}{8}$

$$\therefore \frac{35\cdot 14}{2\cdot 576}S + 1 - \frac{35}{24} + \frac{35\times 14}{2\cdot 576} = (1-x)^{-p/q}$$

$$\frac{245}{576}S + \frac{576 - 840 + 245}{576} = \left(1 - \frac{7}{8}\right)^{35/21}$$

$$\frac{245}{576}S - \frac{19}{576} = \left(\frac{1}{8}\right)^{5/3}$$

$$\therefore S = \frac{576}{245}\left[\left(\frac{1}{8}\right)^{5/3} + \frac{19}{576}\right]$$

Ex. Infinite sum to series $\frac{1}{9\cdot 18} - \frac{1\cdot 3}{9\cdot 18\cdot 27} + \frac{1\cdot 3\cdot 5}{9\cdot 18\cdot 27\cdot 36} - \ldots$

Sol:

Let $S = \frac{1}{9\cdot 18} - \frac{1\cdot 3}{9\cdot 18\cdot 27} + \frac{1\cdot 3\cdot 5}{9\cdot 18\cdot 27\cdot 36} - \ldots$

$$= \frac{1}{\underline{|2}}\cdot\frac{1}{9^2} - \frac{1\cdot 3}{\underline{|3}}\frac{1}{9^3} + \frac{1\cdot 3\cdot 5}{\underline{|4}}\frac{1}{9^4} + \ldots\infty$$

$$(-1)S = \frac{(-1)\cdot 1}{\underline{|2}}\left(\frac{1}{9^2}\right) + \frac{(-1)\,1\cdot 3}{\underline{|3}} + \frac{(-1)\,1\cdot 3\cdot 5}{\underline{|4}}\frac{1}{9^4} + \ldots$$

$$-S + 1 - \frac{(-1)}{1!}\frac{1}{9} = 1 - \frac{(-1)}{1!}\cdot\frac{1}{9} + \frac{(-1)1}{2}\cdot\frac{1}{9^2} - \frac{(-1)\cdot 1\cdot 3\cdot}{3!}\left(\frac{1}{9^3}\right) + \cdots$$

comparing $(1-x)^{-p/q} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p+q}{2!}\left(\frac{x}{q}\right)^2 + \frac{p+2q}{3!}\left(\frac{x}{q}\right)^3 + \cdots\infty$

$$p = -1,\quad q = 2,\quad \frac{x}{q} = \frac{1}{9} \Rightarrow x = \frac{2}{9}$$

$$-S + \frac{10}{9} = (1+x)^{-p/q} = \left(1 + \frac{2}{9}\right)^{1/2}$$

$$= \left(\sqrt{\frac{11}{9}}\right)^{1/2}$$

$$-S = \sqrt{\frac{11}{9}} - \frac{10}{9}$$

$$\therefore S = \frac{10}{9} - \sqrt{\frac{11}{9}}$$

$$S = \frac{10}{9} - \frac{1}{3}\sqrt{11}$$

Q.4 Sum to ∞ The series $2 + \sum_{n=1}^{\infty} \frac{1}{3}\frac{1}{6^{n-1}}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$

Sol:

$$S = 2 + \sum_{n=1}^{\infty} \frac{1}{3}\frac{1}{6^{n-1}}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$$

$$= 2 + \frac{1}{3}\left[\frac{1}{1!} + \frac{1\cdot 3}{2!}\frac{1}{6} + \frac{1\cdot 3\cdot 5}{3!}\frac{1}{6^2} + \cdots\infty\right]$$

$$= 2 + \frac{6^2}{3}\left[\frac{1}{1!}\cdot\frac{1}{6} + \frac{1\cdot 3}{2}\frac{1}{6^2} + \frac{1\cdot 3\cdot 5}{3!}\frac{1}{6^3} + \cdots\infty\right]$$

$$= 2 + 2\left[\frac{1}{1!}\cdot\frac{1}{6} + \frac{1\cdot 3}{2}\frac{1}{6^2} + \frac{1\cdot 3\cdot 5}{3!} + \frac{1}{6^3} + \cdots\infty\right]$$

$$s = 2\left[1 + \frac{1}{\lfloor 1}\cdot\frac{1}{6} + \frac{1\cdot 3}{\lfloor 2}\frac{1}{6^2} + \frac{1\cdot 3\cdot 5}{\lfloor 3}\cdot\frac{1}{6^3} + \cdots\right]$$

Here $p = 1$, $q = 2$, $\frac{x}{q} = \frac{1}{6} \Rightarrow x = \frac{1}{3}$

$$s = 2(1-x)^{-p/q}$$

$$= 2\left(1-\frac{1}{3}\right)^{-1/2}$$

$$= 2\left(\frac{2}{3}\right)^{-1/2}$$

$$= 2\left(\frac{3}{2}\right)^{1/2}$$

$$= \sqrt{2}\cancel{\sqrt{2}} \times \frac{\sqrt{3}}{\cancel{\sqrt{2}}}$$

$$\boxed{S = \sqrt{6}}$$

Ex 9 Show that $1 - \frac{n+x}{\lfloor 1 (1+x)} + \frac{(n+2x)(n-1)}{\lfloor 2 (1+x)^2} - \frac{(n+3x)(n-1)(n-2)}{\lfloor 3 (1+x)^3} + \cdots \infty = 0$

Sol:

$$LHS = 1 - \frac{n+x}{\lfloor 1 (1+x)} + \frac{(n+2x)(n-1)}{\lfloor 2 (1+x)^2} - \frac{(n+3x)(n-1)(n-2)}{\lfloor 3 (1+x)^3} + \cdots \infty$$

$$= \left[1 - \frac{n}{\lfloor 1 (1+x)} + \frac{n(n-1)}{\lfloor 2 (1+x)^2} - \frac{n(n-1)(n-2)}{\lfloor 3 (1+x)^3} + \cdots \infty\right]$$

$$+ \left[\frac{-x}{\lfloor 1 (1+x)} + \frac{2x(n-1)}{\lfloor 2 (1+x)^2} - \frac{3x(n-1)(n-2)}{\lfloor 3 (1+x)^3} + \cdots\right]$$

$$= \left(1 - \frac{1}{1+x}\right)^n - \frac{x}{1+x}\left[1 - \frac{\cancel{2}(n-1)}{\cancel{2}\times\lfloor 1 (1+x)} + \frac{\cancel{3}(n-1)(n-2)}{\cancel{3}\times\lfloor 2 (1+x)^2} + \cdots\right]$$

$$= \left(1 - \frac{1}{1+x}\right)^n - \frac{x}{1+x}\left[1 - \frac{(n-1)}{\lfloor 1}\frac{1}{(1+x)} + \frac{(n-1)(n-2)}{\lfloor 2 (1+x)^2} + \cdots\right]$$

$$= \left(1-\frac{1}{1+x}\right)^n - \frac{x}{1+x}\left(1-\frac{1}{1+x}\right)^{n-1}$$

$$= \left(\frac{x}{1+x}\right)^n - \frac{x}{1+x}\left(\frac{x}{1+x}\right)^{n-1}$$

$$= \left(\frac{x}{1+x}\right)^n - \left(\frac{x}{1+x}\right)^n$$

$$= 0$$

LHS = RHS ,,

Ex 10 Show that $x > -1/2$,

$$\frac{x}{\sqrt{1+x}} \sum_{n=1}^{\infty} \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n}\left(\frac{x}{1+x}\right)^{n+1} + \frac{x}{1+x}$$

Sol:

Let $S = \sum \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4 \cdots 2n}\left(\frac{x}{x+1}\right)^{n+1} + \frac{x}{x+1}$

$$= \left(\frac{x}{1+x}\right)\sum \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}\cdot\left(\frac{x}{x+1}\right)^n + \frac{x}{1+x}$$

$$= \frac{x}{1+x}\left[1 + \frac{1}{2}\,\frac{x}{x+1} + \frac{1}{2}\cdot\frac{3}{4}\cdot\left(\frac{x}{x+1}\right)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{x}{x+1}\right)^3 + \cdots\right]$$

$$= \frac{x}{1+x}\left[1 + \frac{1}{\lfloor 1}\,\frac{x}{2(x+1)} + \frac{1\cdot 3}{\lfloor 2}\cdot\left(\frac{x}{2(x+1)}\right)^2 + \frac{1\cdot 3\cdot 5}{\lfloor 3}\left(\frac{x}{x+1}\right)^3 + \cdots \infty\right]$$

Here $p = 1$

$q = 2$

$\frac{X}{Q} = \frac{x}{2(x+1)}$ $\quad\therefore X = \frac{x}{x+1}$

$$S = \frac{x}{x+1}\left[1-x\right]^{-p/q}$$

$$= \frac{x}{x+1}\left(1-\frac{x}{x+1}\right)^{-1/2}$$

$$= \frac{x}{x+1}\sqrt{x+1} = \frac{x}{\sqrt{x+1}}$$

Ex 11 Assuming that the square and higher powers of x be neglected $\frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = 4-\frac{19x}{3}$

$$\frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = \frac{(1+x)^{1/2}(4)^{3/2}\left(1-\frac{3x}{4}\right)^{3/2}}{(8)^{1/3}\left(1+\frac{5x}{8}\right)^{1/3}}$$

$$= \frac{(1+x)^{1/2}(2^2)^{3/2}\left(1-\frac{3x}{4}\right)^{3/2}}{(2^3)^{1/3}\left(1+\frac{5x}{8}\right)^{1/3}}$$

$$= \frac{(1+x)^{1/2}\,\cancel{2}^{3}\left(1-\frac{3x}{4}\right)^{3/2}}{\cancel{2}\left(1+\frac{5x}{8}\right)^{1/3}}$$

$$= 4(1+x)^{1/2}\left(1-\frac{3x}{4}\right)^{3/2}\left(1+\frac{5x}{8}\right)^{-1/3}$$

$$= 4\left[1+\frac{1}{2}x\right)\left(1-\frac{3}{2}\left(\frac{3x}{4}\right)\right)\left(1-\frac{1}{3}\cdot\frac{5x}{8}\right)$$

$$= 4\left(1+\frac{x}{2}\right)\left(1-\frac{9x}{8}\right)\left(1-\frac{5x}{24}\right)$$

$$= 4+2x-\frac{9x}{2}-\frac{5x}{6}$$

$$= 4-\frac{19x}{3}$$

Ex 12 If x be so small that the squares and higher powers of x may be neglected prove that

$$\frac{(9+2x)^{1/2}(3+4x)}{\sqrt{1-x}} = 9 + \frac{74x}{5} \text{ nearly}$$

Sol: Here $\frac{(9-2x)^{1/2}(3+4x)}{\sqrt[5]{1-x}} = (9 + \frac{2x}{9})^{1/2}(3+4x)(1-x)^{-1/5}$

$$= (3^2)^{1/2}\left(1+\frac{2x}{9}\right)^{1/2}(3+4x)(1-x)^{-1/5}$$

$$= 3\left(1+\frac{2x}{9}\right)^{1/2}(3+4x)(1-x)^{-1/5}$$

$$= 3\left(1+\frac{1/2}{1!}\frac{2x}{9}\right)(3+4x)\left(1+\frac{x}{5}\right)$$

$$= 3\left(1+\frac{x}{9}\right)(3+4x)\left(1+\frac{x}{5}\right)$$

$$= 3\left(1+\frac{x}{9}\right)\left(3+\frac{3x}{5}+4x\right)$$

$$= 3\left(3+\frac{3x}{5}+4x+\frac{3x}{9}\right)$$

$$= 9+\frac{9x}{5}+12x+x$$

$$= 9+\frac{74x}{5}$$

Ex 13 If x be so small that x^4 and higher powers may be neglected show that

$$\frac{(1+x+x^2)(1+x)^2}{1-x+x^2} = 1+4x+7x^2+6x^3$$

(iv) $LHS = \frac{(1+x+x^2)(1+2x+x^2)}{1-x+x^2}$

$= (1+2x+x^2+x+2x^2+x^3+x^2+2x^3)(1-x+x^2)^{-1}$

$= (1+3x+4x^2+3x^3)[1-(x-x^2)]^{-1}$

$= (1+3x+4x^2+3x^3)[1+(x-x^2)+(x-x^2)^2+(x-x^2)^3 \ldots]$

$= (1+3x+4x^2+3x^3)(1+x-x^2+\cancel{x^2}+x-2x^3+x^3)$

$= (1+3x+4x^2+3x^3)(1+x-x^3)$

$= 1+x-x^3+3x+3x^2+4x^2+4x^3+3x^3$

$= 1+4x+7x^2+6x^3$

Ex 11 If c be small in comparison with l show that

$\left(\frac{l}{l+c}\right)^{1/2}+\left(\frac{l}{l-c}\right)^{1/2} = 2+\frac{c^2}{4l^2}$ approximately

Sol:

$\left(\frac{l}{l+c}\right)^{1/2}+\left(\frac{l}{l-c}\right)^{1/2} = \left(\frac{l+c}{l}\right)^{-1/2}+\left(\frac{l-c}{l}\right)^{-1/2}$

$= \left(1+\frac{c}{l}\right)^{-1/2}+\left(1-\frac{c}{l}\right)^{-1/2}$

$= \left[1-\frac{1/2}{1}\cdot\frac{c}{l}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{\underline{|2}}\cdot\frac{c^2}{l^2}\right]$

$$+\left[1+\frac{1}{2}\cdot\frac{c}{l}+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{\lfloor 2}\frac{c^2}{l^2}\right]$$

$$=\left(1-\frac{c}{2l}+\frac{3c^2}{8l^2}\right)+\left(1+\frac{c}{2l}+\frac{3c^2}{8l^2}\right)$$

$$=2+\frac{3c^2}{4l^2}$$

Ex 15 If $p-q$ is small compared to p or q show that $\sqrt[n]{\frac{p}{q}}=\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}$ approximately. Find the seventh root of $\frac{131}{133}$

Sol:

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\frac{np+p+nq-q}{pn-p+qn+q}$$

$$=\frac{n(p+q)+(p-q)}{n(p+q)-(p-q)}$$

$$=\frac{\cancel{n(p+q)}\left(1+\frac{p-q}{n(p+q)}\right)}{\cancel{n(p+q)}\left(1-\frac{(p-q)}{n(p+q)}\right)}$$

$$=\frac{1+\frac{p-q}{n(p+q)}}{1-\frac{(p-q)}{n(p+q)}}$$

$$=\frac{\left(1+\frac{p-q}{p+q}\right)^{1/n}}{\left(1-\frac{p-q}{p+q}\right)^{1/n}}$$

$$= \left[\frac{p+q+(p-q)}{p+q-(p-q)}\right]^{1/n}$$

$$= \left(\frac{p}{q}\right)^{1/n}$$

$$\therefore \left(\frac{p}{q}\right)^{n} = \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}$$

Put $n=7$, $p=131$, $q=133$

$$\therefore \sqrt[7]{\frac{131}{133}} = \frac{(7+1)131+(7-1)133}{(7-1)131+(7+1)133}$$

$$= \frac{923}{925}$$

Ex 16 If x is large show that $\sqrt{x^2+4} - \sqrt{x^2+1} = \frac{3}{2x} - \frac{15}{8x^3} + \dots$

Solution:

$$\sqrt{x^2+4} - \sqrt{x^2+1} = x\sqrt{1+\frac{4}{x^2}} - x\sqrt{1+\frac{1}{x^2}}$$

$$= x\left(1+\frac{4}{x^2}\right)^{1/2} - x\left(1+\frac{1}{x^2}\right)^{1/2}$$

$$x\left(1+\frac{4}{x^2}\right)^{1/2} = x\left[1+\frac{1}{2}\left(\frac{4}{x^2}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{\lfloor 2}\frac{16}{x^4}+\dots\right]$$

$$= x\left[1+\frac{2}{x^2}-\frac{16}{8x^4}+\frac{8}{\dots}\right]$$

$$= x\left[1+\frac{2}{x^2}-\frac{2}{x^4}+\frac{4}{x^6}-\dots\right]$$

$$x\left(1+\frac{4}{x^2}\right)^{1/2} = x+\frac{2}{x}-\frac{2}{x^3}+\frac{4}{x^5}$$

$x\left(1+\frac{1}{x^2}\right)^{1/2} = x\left[1+\frac{1}{2x^2}+\frac{\frac{1}{2}(-\frac{1}{2})}{\lfloor 2}\frac{1}{x^4}+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{\lfloor 3}\frac{1}{x^6}\right]$

$= x+\frac{1}{2x}-\frac{1}{8x^3}+\frac{1}{16x^5} \longrightarrow ②$

Subtracting ② and ①

$\therefore x\left(1+\frac{4}{x^2}\right)^{1/2} - x\left(1+\frac{1}{x^2}\right) = \frac{3}{2x}-\frac{15}{8x^3}+\frac{63}{16x^5}$

15. If x and y are small show that $\frac{(1+y)^x}{(1+x)^y} = 1+\frac{1}{2}xy(x-y)$

Sol:

$\frac{(1+y)^x}{(1+x)^y} = (1+y)^x(1+x)^{-y}$

$= \left[1+xy+x\frac{(x-1)}{2}y^2+\cdots\right]\left[1-yx+\frac{1}{2}y(y+1)x^2 \cdots\right]$

$= 1+xy+x\frac{(x-1)y^2}{2}-yx-x^2y^2+\frac{y(y+1)}{2}x^2+\cdots$

$= 1+\frac{x^2y^2}{2}-\frac{xy^2}{2}-x^2y^2+\frac{y^2x^2}{2}+\frac{yx^2}{2}+\cdots$

$= 1+\frac{xy}{2}(x-y)$

Ex Find the sum of the first r coefficient in the expansion of $(1-x)^{-3}$

Sol: Let $f(x) = (1-x)^{-3}$

The sum of the first r coefficients in the expansion of $(1-x)^{-3}$

$=$ coefficient of x^r in $\frac{(1-x)^{-3}}{1-x}$

$=$ coefficient of x^r in $(1-x)^{-4}$

$$= \frac{(r+1)(r+2)(r+3)}{6}$$

Example 19

If x is small so that x^3, x^4 and higher powers of x can be neglected show that nth root of $(1+x)$

$$= \frac{2n+(n+1)x}{2n+(n-1)x} \text{ nearly}$$

Sol:

$$\frac{2n+(n+1)x}{2n+(n-1)x} = \frac{2n\left[1+\frac{(n+1)}{2n}x\right]}{2n\left[1+\frac{(n-1)}{2n}x\right]}$$

$$= 2n\left[1+\frac{(n+1)}{2n}x\right]\left[1+\frac{(n-1)}{2n}x\right]^{-1}$$

$$= \left(1+\frac{(n+1)}{2n}x\right)\left(1-\frac{(n-1)}{2n}x\right)$$

$$= 1+\frac{n+1}{2n}x - \frac{n-1}{2n}x$$

$$= 1+\frac{(n+1)-(n-1)}{2n}x$$

$$= 1+\frac{\cancel{n}+1-\cancel{n}+1}{2n}x$$

$$= 1+\frac{2}{2n}x$$

$$= 1+\frac{1}{n}x$$

$$= (1+x)^{1/n} \text{ nearly}$$

The field strength H due to a magnet of length $2l$ and moment M at a point on its axis distant x from the centre is $\frac{M}{2l}\left[\frac{1}{(x-l)^2}-\frac{1}{(x+l)^2}\right]$. Show that if x is large compared to l then H is approximately equal to $\frac{2M}{x^3}$

Sol:

$$\frac{M}{2l}\left[\frac{1}{(x-l)^2}-\frac{1}{(x+l)^2}\right]$$

$$= \frac{M}{2lx^2}\left[\frac{1}{(1-\frac{l}{x})^2}-\frac{1}{(1+\frac{l}{x})^2}\right]$$

$$= \frac{M}{2lx^2}\left[\left(1-\frac{l}{x}\right)^{-2}-\left(1+\frac{l}{x}\right)^{-2}\right]$$

$$= \frac{M}{2lx^2}\left[1+\frac{2l}{x}+\frac{3l^2}{x^2}-\left(1-\frac{2l}{x}+\frac{3l^2}{x^2}\right)\right]$$

$$\Rightarrow \frac{2M}{x^3} \text{ nearly if } \frac{l}{x} \text{ is very small.}$$