$\therefore L(x,\lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1 g_1(x) - \lambda_2 g_2(x)$.

$L(x,\lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$ $\rightarrow (3)$

Diff (3) p.w.r.t $x_1, x_2, x_3, \lambda_1, \lambda_2$ and equate to zero.

$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2$, $\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - \lambda_1 - 5\lambda_2 = 0 \rightarrow (4)$

$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2$, $\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - \lambda_1 - 2\lambda_2 = 0 \rightarrow (5)$

$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2$, $\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - 3\lambda_1 - \lambda_2 = 0 \rightarrow (6)$

$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2)$. $\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 + 3x_3 - 2 = 0 \rightarrow (7)$

$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5)$, $\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 5x_1 + 2x_2 + x_3 - 5 = 0$ $\rightarrow (8)$

Solve (4) and (5).

$\lambda_1 + 5\lambda_2 = 2x_1$

$$
\begin{aligned}
2x_1 - \lambda_1 - 5\lambda_2 &= 0 \\
2x_2 - \lambda_1 - 2\lambda_2 &= 0 \\
(-) \quad (+) \quad (+) & \\
\hline
2x_1 - 2x_2 - 3\lambda_2 &= 0 \rightarrow (9)
\end{aligned}
$$

Solve 5 and (6).

$$
\begin{aligned}
2x_2 - \lambda_1 - 2\lambda_2 &= 0 \\
2x_3 - 3\lambda_1 - \lambda_2 &
\end{aligned}
$$

$(4) \times 3 \Rightarrow 6x_1 - \cancel{3\lambda_1} + 15\lambda_2 = 0$

$(6) \times 1 \Rightarrow 2x_3 - \cancel{3\lambda_1} - \lambda_2 = 0$

$\quad (-) \quad (+) \quad (+)$

$4x_3 - 4\lambda_2 = 0.$

$x_3 - \lambda_2 = 0.$ $\boxed{x_3 = \lambda_2}$

$6x_1 - 2x_3 - 14\lambda_2 = 0 \rightarrow (10).$

$(5) \times 3 \Rightarrow 6x_2 - \cancel{3\lambda_1} - 6\lambda_2 = 0$

$(6) \times 1 \Rightarrow 2x_3 - \cancel{3\lambda_1} - \lambda_2 = 0$

$\quad (-) \quad (+) \quad (+)$

$6x_2 - 2x_3 - 5\lambda_2 = 0 \rightarrow (11).$

$(10) \times 5 \Rightarrow 30x_1 - 10x_3 - \cancel{70\lambda_2} = 0$

$(11) \times 14 \Rightarrow 84x_2 - 28x_3 - \cancel{70\lambda_2} = 0$

$\quad (-) \quad (+) \quad (+)$

$\cancel{46x}30x_1 - 84x_2 + 18x_3 = 0$

$\Rightarrow 30x_1 - 84x_2 + 18x_3 = 0. \rightarrow (12).$

$(8) \times 6 \Rightarrow \cancel{30x_1} + 12x_2 + 6x_3 = 30$

$\cancel{30x_1} - 84x_2 + 18x_3 = 0.$

$\quad (-) \quad (+) \quad (-) \quad (-)$

$96x_2 - 12x_3 = 30. \rightarrow (13)$

$(7) \times 5 \Rightarrow \cancel{5x_1} + 5x_2 + 15x_3 = 10.$

$(8) \times 1 \Rightarrow \cancel{5x_1} + 2x_2 + x_3 = 5$

$\quad (-) \quad (-) \quad (-) \quad (-)$

$3x_2 + 14x_3 = 5. \rightarrow (14).$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 \qquad \frac{\partial^2 L}{\partial x_1^2} = 2 \qquad \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0.$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2. \qquad \frac{\partial^2 L}{\partial x_2^2} = 2. \qquad \frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 \qquad \frac{\partial^2 L}{\partial x_3^2} = 2. \qquad \frac{\partial^2 L}{\partial x_3 \partial x_1} = 0$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_3} = 0. \qquad \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0 \qquad \frac{\partial^2 L}{\partial x_3 \partial x_1} = 0.$$

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$H = \left[\frac{\partial g_i(x)}{\partial x_j} \right]_{m \times n} \Rightarrow 2 \times 3.$$

$$H = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix}$$

$$g_1 = x_1 + x_2 + 3x_3 - 2. \qquad \frac{\partial g_1}{\partial x_1} = 1. \quad \frac{\partial g_1}{\partial x_2} = 1 \quad \frac{\partial g_1}{\partial x_3} = 3.$$

$$g_2 = 5x_1 + 2x_2 + x_3 - 5$$

$$\frac{\partial g_2}{\partial x_1} = 5 \qquad \frac{\partial g_2}{\partial x_2} = 2 \qquad \frac{\partial g_2}{\partial x_3} = 1.$$

$$H = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix}.$$

$$H^T = \begin{bmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} Q & H^T \\ H & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 1 & 1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \end{bmatrix}$$

Objective is to minimize.

Starting with the order $(2m+1)$.

last $(n-m)$ principal minors are same sign with the sign $(-1)^m$.

Here $m=2$.

order $2m+1 = 2\cdot 2+1 = 5$. $n-m = 3-2 = 1$.

5×5 order. $(-1)^m = (-1)^2 = +$ve sign.

$$\begin{vmatrix} 2 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 1 & 1 & 3 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 1 & 1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{array}{c} \begin{matrix} + & - & + & - & + \end{matrix} \\ \begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & 1 & -3 \\ -6 & 0 & 2 & 3 & -14 \\ 1 & 1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \end{vmatrix} \end{array}$$

Here $C_1 \to C_1 - 2C_4$.
$C_5 \to C_5 - 5C_4$.

$C_1 \to C_1 + C_4$

$$= -1 \begin{vmatrix} -2 & 2 & 0 & -3 \\ -6 & 0 & 2 & -14 \\ 1 & 1 & 3 & 0 \\ 5 & 2 & 1 & 0 \end{vmatrix}$$

$$= -1 \left(-(-3) \begin{vmatrix} -6 & 0 & 2 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{vmatrix} + (-14) \begin{vmatrix} -2 & 2 & 0 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{vmatrix} \right]$$

$$= -1 \left(3[-6(1-6) + 2(2-5)] - 14 [-2(1-6) - 2(1-15)] \right.$$

$= -1(3(-6+36)+4-10] - 14[-2+12-2+30]$

$= -1[-4+52] = -14[48]$

$= -1[84] + 212$

$= -1(-460)$

$|D| = 460 = +ve$ sign.

$\therefore$ given function is minimum.

$x_1 = \frac{37}{46}$ $x_2 = \frac{8}{23}$ $x_3 = \frac{13}{46}$.

$Z = x_1^2 + x_2^2 + x_3^2$

$= \left(\frac{37}{46}\right)^2 + \left(\frac{8}{23}\right)^2 + \left(\frac{13}{46}\right)^2$

$= \frac{1369 + 256 + 169}{2116}$

$= \frac{1794}{2116}$

$$\boxed{\min z = \frac{897}{1058}}$$