Proof: To prove $f+g$ is continuous at $a \in M$

(ie) To prove $\lim_{x \to a} (f+g)(x) = (f+g)(a)$.

ie) To prove, given $\varepsilon_1 > 0$ there exists $\delta > 0$ such that $|(f+g)(x) - (f+g)(a)| < \varepsilon_1$ when $\rho(x,a) < \delta$.

Given f is continuous at 'a' and g is continuous at 'a'

(ie) $\lim_{x \to a} f(x) = f(a)$

By defn given $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that

$$|f(x) - f(a)| < \varepsilon_1/2 \text{ when } \rho(x,a) < \delta_1 \quad \text{--- (1)}$$

Also given $\lim_{x \to a} g(x) = g(a)$

By defn given $\varepsilon_1 > 0$ there exists $\delta_2 > 0$ such that

$$|g(x) - g(a)| < \varepsilon_1/2 \text{ when } \rho(x,a) < \delta_2 \quad \text{--- (2)}$$

Let $\delta = \min\{\delta_1, \delta_2\}$

$$\Rightarrow \text{when } \rho(x,a) < \delta \text{ then } \left.\begin{array}{l} |f(x) - f(a)| < \varepsilon_1/2 \\ \text{and } |g(x) - g(a)| < \frac{\varepsilon_1}{2} \end{array}\right\} \quad \text{--- (3)}$$

when $\rho(x,a) < \delta$

Consider

$$\begin{aligned} |(f+g)(x) - (f+g)(a)| &= |(f(x)+g(x)) - (f(a)+g(a))| \\ &= |(f(x) - f(a)) + g(x) - g(a)| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &< \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} \end{aligned}$$

$$|(f+g)(x) - (f+g)(a)| < \varepsilon_1 \text{ when } \rho(x,a) < \delta$$

$\Rightarrow \lim_{x \to a} (f+g)(x) = (f+g)(a)$ $\therefore f+g$ is continuous at $a \in M$

To prove $(f-g)$ is continuous at $a \in M$.

(i.e.) To prove $\lim_{x \to a} (f-g)(x) = (f-g)(a)$.

(i.e.) To prove $|(f-g)(x) - (f-g)(a)| < \varepsilon$ when $\rho(x,a) < \delta$.

Proof: using (3)

when $\rho(x,a) < \delta$

$$|f(x) - f(a)| < \frac{\varepsilon}{2}, \quad |g(x) - g(a)| < \varepsilon/2 \quad \text{—} \quad (3)$$

when $\rho(x,a) < \delta$

consider

$$|(f-g)(x) - (f-g)(a)| = |f(x) - g(x) - f(a) + g(a)|$$

$$= |(f(x) - f(a)) + (g(a) - g(x))|$$

$$\leq |f(x) - f(a)| + |g(a) - g(x)|$$

$$\leq |f(x) - f(a)| + |g(x) - g(a)|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$|(f-g)(x) - (f-g)(a)| < \varepsilon \quad \text{when } \rho(x,a) < \delta$$

$$\Rightarrow \lim_{x \to a} (f-g)(x) = (f-g)(a)$$

$\Rightarrow (f-g)$ is continuous at $a \in M$.

Proof(c) To prove fg is continuous at 'a'

(i.e.) To prove $\lim_{x \to a} (fg)(x) = (fg)(a)$.

(i.e.) To prove $|(fg)(x) - (fg)(a)| < \varepsilon$ when $\rho(x,a) < \delta$

Proof given g is continuous at 'a'

(i.e.) $\lim_{x \to a} g(x) = g(a)$ $\qquad$ let $g(a) = N$

$$\lim_{x \to a} g(x) = N$$

By defn given $\varepsilon_1 > 0$ there exists $\delta_1 > 0$

$|g(x) - N| < \varepsilon_1$ when $\rho_*(x, a) < \delta_1$

$|g(x) - N| < 1$ when $\rho(x, a) < \delta_1$

$\Rightarrow |g(x)| < |N| + 1$ when $\rho(x, a) < \delta_1$

$|g(x)| < Q$ when $\rho(x, a) < \delta_1$ —— (4)

Also given $\lim_{x \to a} f(x) = f(a)$ let $f(a) = L$

$\lim_{x \to a} f(x) = L$

(a) given $\varepsilon_1 > 0$ $\exists$ $\delta_2 > 0$

$|f(x) - L| < \frac{\varepsilon_1}{2 \cdot Q}$ when $\rho_*(x, a) < \delta_2$ —— (5)

and there exists $\delta_3 > 0$

$|g(x) - N| < \frac{\varepsilon_1}{2|L|}$ when $\rho_*(x, a) < \delta_3$ —— (6)

let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$

$\therefore$ when $\rho(x, a) < \delta$ (4), (5) and (6) are true

$\therefore$ consider

$$|f(x)g(x) - f(a)g(a)| = |f(x)g(x) - LN|$$

$$= |f(x)g(x) - Lg(x) + Lg(x) - LN|$$

$$= |g(x)(f(x) - L) + L(g(x) - N)|$$

$$\leq |g(x)|\,|f(x) - L| + |L|\,|g(x) - N|$$

$$\leq Q \cdot \frac{\varepsilon_1}{2 \cdot Q} + |L| \frac{\varepsilon_1}{2|L|}$$

$|f(x)g(x) - f(a)g(a)| < \varepsilon_1$ when $\rho(x, a) < \delta$

$|(fg)(x) - (fg)(a)| < \varepsilon_1$ when $\rho(x, a) < \delta$.

$\Rightarrow \lim_{x \to a} (fg)(x) = (fg)(a)$

$\Rightarrow$ fg is continuous at 'a'

To prove $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \left(\frac{f}{g}\right)(a)$ provided $g(a) \neq 0$

To prove $\left|\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(a)\right| < \varepsilon_1$ when $\rho(x, a) < \delta$

$\left|\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}\right| < \varepsilon_1$ when $\rho(x, a) < \delta$.

(i.e) T.P $\left|\frac{f(x)g(a) - g(x)f(a)}{g(x)g(a)}\right| < \varepsilon_1$ when $\rho(x, a) < \delta$

Proof