whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in M_1, ⑧

converging to a.

$\Rightarrow \lim_{n\to\infty} x_n = a$

By defn, given $\delta > 0$ there exists $N \in I$, such that $x_n \in B[a;\delta]$ when $n \geq N$

$\Rightarrow$ when $n \geq N$, $x_n \in B[a;\delta]$ using ①

$$f(x_n) \in B[f(a);\varepsilon_1]$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$$

$\Rightarrow$ The sequence $\{f(x_n)\}_{n=1}^{\infty}$ of points in M_2 converges to $f(a)$.

hence (c) proved.

Conversely Suppose (a) is true

(i) Given $\varepsilon_1 > 0$ there exists $\delta > 0$ such that

$\rho_2(f(x), f(a)) < \varepsilon_1$ when $(\rho_1(x,a) < \delta)$

$$\Rightarrow \lim_{x\to a} f(x) = f(a)$$

$\Rightarrow f$ is continuous at 'a'.

Suppose (b) is true

$$f^{-1}(B[f(a);\varepsilon_1]) \supset B[a;\delta]$$

(i) when $x \in B[a;\delta]$ then $x \in f^{-1}[B[f(a);\varepsilon_1]]$

$\Rightarrow$ (i) when $\rho_1(x,a) < \delta$ then $f(x) \in B[f(a);\varepsilon_1]$

$\Rightarrow$ when $\rho_1(x,a) < \delta$ then $\rho_2(f(x);f(a)) < \varepsilon_1$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$

$\Rightarrow f$ is continuous at 'a'.

Suppose (C) is true

when $\lim_{n \to \infty} x_n = a$ then $\lim_{n \to \infty} f(x_n) = f(a)$

$\Rightarrow$ when $n \geq N$, $\rho_1(x_n, a) < \delta$

then $\rho_2(f(x_n), f(a)) < \epsilon$

$\Rightarrow$ when $x_n \in B[a; \delta]$ then $f(x_n) \in B[f(a); \epsilon]$

$\Rightarrow$ when $x \in B[a; \delta]$ then $f(x) \in B[f(a); \epsilon]$

$\Rightarrow$ when $\rho_1(x, a) < \delta$ then $\rho_2(f(x), f(a)) < \epsilon$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$

$\Rightarrow f$ is continuous at 'a'

Theorem 5.3D : Let $\langle M_1, \rho_1 \rangle, \langle M_2, \rho_2 \rangle, \langle M_3, \rho_3 \rangle$ be metric spaces and let $f: M_1 \to M_2$, $g: M_2 \to M_3$. If 'f' is continuous at $a \in M_1$ and g is continuous at $f(a) \in M_2$ then $g \circ f$ is continuous at 'a'.

Proof To prove $\lim_{x \to a} (g \circ f) x = (g \circ f)(a)$

(e) To prove $\lim_{x \to a} g(f(x)) = g(f(a))$

(e) To prove given $\epsilon > 0$, we must find $\delta > 0$ such that $\rho_3(g(f(x)), g(f(a))) < \epsilon$ when $\rho_1(x, a) < \delta$

Proof Given g is continuous at $f(a)$

Let $b = f(a)$. Now by hypothesis $\lim_{y \to b} g(y) = g(b)$

$\therefore$ given $\varepsilon_1 > 0$ there exists $\eta > 0$ such that

$$\rho_3(g(y), g(b)) < \varepsilon_1 \text{ when } \rho_2(y, b) < \eta \quad \text{--- } (1)$$

also given f is continuous at 'a'

$$\lim_{x \to a} f(x) = f(a)$$

$\Rightarrow$ given $\eta > 0$ there exists $\delta > 0$ such that

$$\rho_2(f(x), f(a)) < \eta \text{ when } \rho_1(x, a) < \delta$$

Here $y = f(x)$ $\quad b = f(a)$

$\Rightarrow$ $\rho_2(y, b) < \eta$ when $\rho_1(x, a) < \delta$

using (1)

$\Rightarrow$ when $\rho_1(x, a) < \delta$ then $\rho_3(g(y), g(b)) < \varepsilon_1$

$\Rightarrow$ when $\rho_1(x, a) < \delta$ then $\rho_3(g(f(x)), g(f(a))) < \varepsilon_1$

$$\Rightarrow \quad \lim_{x \to a} g(f(x)) = g(f(a))$$

$$\Rightarrow \quad \lim_{x \to a} (g \circ f)(x) = (g \circ f)(a)$$

$\Rightarrow$ $g \circ f$ is continuous at 'a'.

Theorem 5.3E. Let M be a metric space and let f and g be real-valued functions which are continuous at $a \in M$. Then $f+g$, $f-g$, and fg are also continuous at a. Furthermore, if $g(a) \neq 0$, then $\left(\frac{f}{g}\right)$ is continuous at 'a'.