5·1 C Theorem: If f and g are real-valued functions, if f is continuous at a, and if g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Proof we must prove that $\lim_{x \to a} (g \circ f)(x) = (g \circ f)(a)$ or

$$\lim_{x \to a} g(f(x)) = g(f(a))$$

(e) To prove, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|g(f(x)) - g(f(a))| < \varepsilon \qquad (0 < |x-a| < \delta) \quad \text{—} (1)$$

Given g is continuous at $f(a)$

let $b = f(a)$, Now by hypothesis

$\lim_{y \to b} g(y) = g(b)$. Hence there exists

$\eta > 0$ such that

$$|g(y) - g(b)| < \varepsilon \quad |y - b| < \eta \quad \text{—} (2)$$

Also given f is continuous at 'a' i.e. $\lim_{x \to a} f(x) = f(a)$

given $\eta > 0$ $\exists$ $\delta > 0$ such that

$$|f(x) - f(a)| < \eta. \quad |x-a| < \delta \quad \text{——} (3)$$

Thus if $|x-a| < \delta$, then $|f(x) - f(a)| < \eta$

using (2) $|g(f(x)) - g(f(a))| < \varepsilon$

$\Rightarrow$ $|g(f(x)) - g(f(a))| < \varepsilon$ when $|x-a| < \delta$

$\Rightarrow$ $\lim_{x \to a} g(f(x)) = g(f(a))$

$\Rightarrow \lim_{x \to a} (g \circ f)(x) = (g \circ f)(a)$

$\Rightarrow (g \circ f)$ is continuous at 'a'.

5·2 Reformulation:

We have defined "f is continuous at a" to mean $\lim_{x \to a} f(x) = f(a)$. That is f is continuous at 'a' if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ if $0 < |x-a| < \delta$ this is the Reformulation definition.

Theorem 5·2 A:

The real valued function f is continuous at $a \in R^1$ if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ $(|x-a| < \delta)$

Proof Let f is continuous at $a \in R^1$

ie) $\lim_{x \to a} f(x) = f(a)$

By defn given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ when $0 < |x-a| < \delta$

Hence proved part (i)

If given $\varepsilon > 0$ there exists $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon$ $|x-a| < \delta$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$

$\Rightarrow$ f is continuous at 'a'

Hence proved part (ii)

In order to give another reformulation of the definition of continuity, we introduce the following definition.

5.2B definition: If $a \in R^1$ and $r > 0$, we define $B[a;r]$ to be the set of all $x \in R^1$, whose distance to a is less than r. That is

$$B[a;r] = \{x \in R^1 \mid |x-a| < r\}$$

we call $B[a;r]$ the open ball of radius 'r' about 'a'.

5.2C Theorem:
The real-valued function f is continuous at $a \in R^1$ if and only if the inverse image under f of any open ball $B[f(a);\varepsilon]$ about $f(a)$ contains an open ball $B[a;\delta]$ about a. (That is given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f^{-1}\left[B[f(a);\varepsilon]\right) \supset B[a;\delta].$$

Proof: part (i)
let f is continuous at $a \in R^1$

ie) $\lim\limits_{x \to a} f(x) = f(a)$

By defn; given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ when $|x - a| < \delta$

$$\Rightarrow \quad f(x) \in B[f(a):\varepsilon] \quad \text{when } x \in B[a;\delta]$$

$$\Rightarrow \quad x \in f^{-1}(B[f(a);\varepsilon]) \text{ when } x \in B[a;\delta]$$

$$\Rightarrow \quad f^{-1}(B[f(a);\varepsilon]) \supset B[a;\delta]$$

$\Rightarrow$ The inverse image under f of any open ball $B[f(a);\varepsilon]$

about $f(a)$ contains an open ball $B[a;\delta]$ about 'a'.

[when $x \in A$, then $x \in B$ $\Rightarrow A \subset B$]

Conversely

if $f^{-1}[B[f(a);\varepsilon]] \supset B[a;\delta]$

$\Rightarrow$ when $x \in B[a;\delta]$ then $x \in f^{-1}[B[f(a);\varepsilon]]$

$\Rightarrow$ when $|x-a|<\delta$ then $f(x) \in B[f(a);\varepsilon]$

$\Rightarrow$ when $|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$

$\Rightarrow \lim\limits_{x\to a} f(x) = f(a)$

$\Rightarrow f$ is continuous at $a \in R^1$

5.2D Theorem: The real valued function f is continuous at $a \in R^1$ if and only if when ever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converging to $f(a)$.

That is f is continuous at a if and only if $\lim\limits_{n\to\infty} x_n = a$ implies $\lim\limits_{n\to\infty} f(x_n) = f(a)$.