4.1G Defn If f is a real valued function on an interval $J \subset R$, we say that f is nondecreasing on J if $f(x) \le f(y)$ when $(x < y \ \forall x, y \in J)$

we say that f is non increasing on J if

$f(x) \geq f(y)$ when $(x < y \ \forall x, y \in J)$

4.1H Theorem:

Let f be a nondecreasing function on the bounded open interval (a, b). If f is bounded above on (a, b), then $\lim\limits_{x \to b-} f(x)$ exists. Also, If f is bounded below on (a, b) then $\lim\limits_{x \to a+} f(x)$ exist

Proof If f is bounded above on (a, b) and nondecreasing on (a, b).

let $M = \text{l.u.b}\, f(x)$

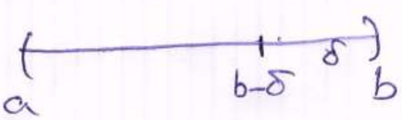
$\therefore f(x) \le M \quad \forall x \in (a, b)$

$f(x) < M + \varepsilon \quad \forall x \in (a, b)$ —— (1)

The number $M - \varepsilon$ is not an upper bound

$\therefore$ there exists $y \in (a, b)$ such that

$f(y) > M - \varepsilon$



let $y = b - \delta$

then $f(b-\delta) = f(y) > M - \varepsilon$

when $b - \delta < x < b$

$f(x) > M - \varepsilon$ since f is increasing function —— (2)

$\therefore$ ~~when~~

$M - \varepsilon < f(x) < M + \varepsilon$ when $b - \delta < x < b$

$\Rightarrow \quad -\varepsilon_1 < f(x) - M < \varepsilon_1 \qquad b-\delta < x < b$

$\Rightarrow \quad |f(x) - M| < \varepsilon_1 \qquad b-\delta < x < b$

$\Rightarrow \quad \lim_{x \to b-} f(x) = M \Rightarrow \lim_{x \to b-} f(x)$ exists.

case(ii) If f is bounded below on (a, b)

let $L = g.l.b\, f(x)$

$\therefore f(x) \geq L \quad \forall \; x \in (a, b)$

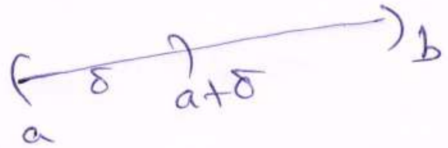
$\Rightarrow \quad f(x) > L - \varepsilon_1 \quad \forall \; x \in (a, b)$ —— ①

since L is the g.l.b of f on (a, b)

$L + \varepsilon_1$ is not an lower bound

$\Rightarrow \; \exists$ atleast one $y \in (a, b)$ such that

$f(y) < L + \varepsilon_1$



let $y = a + \delta$

$\therefore f(a+\delta) = f(y) < L + \varepsilon_1$

$\therefore$ when $a < x < a + \delta$

$f(x) < L + \varepsilon_1$ —— ②

$a < x < a + \delta$

$\therefore \; L - \varepsilon_1 < f(x) < L + \varepsilon_1 \qquad a < x < a + \delta$

$\Rightarrow \quad -\varepsilon_1 < f(x) - L < \varepsilon_1 \qquad a < x < a + \delta$

$|f(x) - L| < \varepsilon_1$

$\Rightarrow \quad \lim_{x \to a+} f(x) = L \quad \therefore \lim_{x \to a+} f(x)$ exists.