UNIT- 4

Sec 3.7 Series whose terms form a nonincreasing sequence.

3.7A Theorem: If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive numbers and if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof Given $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence.

$$\therefore\ a_1 \geq a_2 \geq a_3 \geq \cdots a_n \geq a_{n+1} \geq \cdots$$

we have
$$a_1 \leq a_1 = 2^0 a_{2^0}$$

$$a_2 + a_3 \leq a_2 + a_2 = 2a_2$$

$$a_4 + a_5 + a_6 + a_7 \leq a_4 + a_4 + a_4 + a_4 = 4a_4 = 2^2 a_{2^2}$$

and for any $n \in I$

$$a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1} \leq 2^n a_{2^n}$$

from these inequalities it follows that.

$$\sum_{k=1}^{2^{n+1}-1} a_k \leq \sum_{k=0}^{n} 2^k a_{2^k} \leq \sum_{k=0}^{\infty} 2^k a_{2^k}$$

Hence for any $n \in I$, we have

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=0}^{\infty} 2^k a_{2^k} \quad \text{——— (1)}$$

Given $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges (ie) $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges

using (1) and comparison test

$\sum_{k=1}^{\infty} a_k$ is converges (ie) $\sum_{n=1}^{\infty} a_n$ converges.

3.7B Theorem. If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive numbers and if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. ②

<u>Proof</u> Given $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence.

$$a_1 \geq a_2 \geq a_3 \geq a_4 \cdots a_n \geq a_{n+1} \geq \cdots\cdots$$

$$a_3 + a_4 \geq a_4 + a_4 = 2a_4 = 2a_{2^2} = \frac{1}{2}(2^2 a_{2^2})$$

$$a_5 + a_6 + a_7 + a_8 \geq a_8 + a_8 + a_8 + a_8 = 4a_8 = 2^2 a_{2^3} = \frac{1}{2}(2^3 a_{2^3})$$

and in general

$$a_{2^n+1} + \cdots + a_{2^{n+1}} \geq 2^n a_{2^{n+1}} = \frac{1}{2}(2^{n+1} a_{2^{n+1}})$$

$$\sum_{k=3}^{2^{n+1}} a_k \geq \frac{1}{2} \sum_{k=1}^{n} 2^{k+1} a_{2^{k+1}} = \frac{1}{2} \sum_{k=2}^{n+1} 2^k a_{2^k}$$

as $n \to \infty$

$$\sum_{k=3}^{\infty} a_k \geq \frac{1}{2} \sum_{k=2}^{\infty} 2^k a_{2^k}$$

ie) $$\sum_{n=1}^{\infty} a_n \geq \frac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n} \quad \text{——— (1)}$$

But given $\sum_{n=1}^{\infty} 2^n a_{2^n}$ diverges

$\Rightarrow$ $\frac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n}$ diverges

using (1) and Comparison test

$\sum_{n=1}^{\infty} a_n$ diverges.