So $S_1 \le S_2 \le S_3 \le ---$

$\therefore \{S_n\}_{n=1}^{\infty}$ is increasing sequence.

now To prove seq $\{S_n\}_{n=1}^{\infty}$ is bounded above sequence.

$$S_n < 1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+ - - - + \frac{1}{1\cdot 2\cdot 3 \cdot\cdot n}$$

$$S_n < 1+1+\frac{1}{2}+\frac{1}{1\cdot 2\cdot 2}+ - - - + \frac{1}{1\cdot 2\cdot 2\cdot\cdot\cdot 2}$$

$$S_n < 1+1+\frac{1}{2}+\frac{1}{2^2}+ - - - + \frac{1}{2^{n-1}}$$

$$S_n < 1+1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+ - - - +\left(\frac{1}{2}\right)^{n-1}$$

$$S_n < 1+\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right)$$

$$\because 1+x+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$$

$$S_n < 1+\left(\frac{1-\frac{1}{2^n}}{\frac{1}{2}}\right) \qquad \text{as } n\to\infty \quad \frac{1}{2^n}\to 0$$

as $n \to \infty$

$$S_n < 1+\frac{1}{\frac{1}{2}}$$

$$S_n < 1+2$$

$$S_n < 3 \quad \forall n$$

$\Rightarrow$ Sequence $\{S_n\}_{n=1}^{\infty}$ is bounded above seq

we know that increasing seq and bounded above is convergent.

$\therefore$ The seq $\{S_n\}_{n=1}^{\infty} = \left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ is convergent

$\forall n$

$2 < S_n < 3 \quad \therefore \{S_n\}_{n=1}^{\infty}$ converges to e

ie) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Theorem 2.6D :

A nondecreasing sequence which is not bounded above diverges to infinity.

Proof Given $\{s_n\}_{n=1}^{\infty}$ is nondecreasing.

ie) increasing sequence.

$s_1 \le s_2 \le s_3 \le \cdots \le s_n \le \cdots$ ——— (1)

Also given this seq is not bounded above

∴ Given $M>0$ we must find $N \in I$ ∋

Such that $s_N > M$ ~~$\forall n \ge N$~~

$\Longrightarrow$ ~~$s_M > M$ using (1)~~

$s_n > M \quad \forall n \ge N$ using (1)

$\Longrightarrow$ seq $\{s_n\}_{n=1}^{\infty}$ diverges to ∞.

Theorem 2.6E: A nonincreasing sequence and which is bounded below is convergent.

Let $\{s_n\}_{n=1}^{\infty}$ nonincreasing (ie decreasing)

and bounded below.

$\{s_n\}_{n=1}^{\infty}$ is decreasing seq

$\therefore s_1 \ge s_2 \ge s_3 \ge \cdots \ge s_n \ge s_{n+1} \ge \cdots$ ——— (1)

Also given $\{s_n\}_{n=1}^{\infty}$ bounded below

2.7 Operations on Convergent Sequences.

2.7 A Theorem: If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, If $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$, then

$$\lim_{n\to\infty} (s_n + t_n) = L + M.$$

<u>Proof</u>: To prove given $\epsilon > 0$ there exists $N \in I$ such that $|(s_n+t_n) - (L+M)| < \epsilon$ $(n \geq N)$.

Given $\lim_{n\to\infty} s_n = L$, By defn given $\epsilon > 0$ there exists $N_1 \in I$ such that

$$|s_n - L| < \epsilon/2 \quad \forall n \geq N_1 \quad \text{——} (1)$$

Also given $\lim_{n\to\infty} t_n = M$, By defn, given $\epsilon > 0$ there exists $N_2 \in I$ such that

$$|t_n - M| < \epsilon/2 \quad \forall n \geq N_2 \quad \text{——} (2)$$

Let $N = \max\{N_1, N_2\}$

$$\Rightarrow \forall n \geq N, \text{ then } \left.\begin{array}{l} |s_n - L| < \epsilon/2 \text{ and} \\ |t_n - M| < \epsilon/2 \end{array}\right\} \text{—} (3)$$

$\therefore \forall n \geq N$

Consider
$$\begin{aligned} |(s_n+t_n) - (L+M)| &= |(s_n - L) + (t_n - M)| \\ &\leq |s_n - L| + |t_n - M| \\ &< \epsilon/2 + \epsilon/2 \quad \text{using (3)} \end{aligned}$$

$$|(s_n+t_n) - (L+M)| < \epsilon \quad \forall n \geq N$$

$$\Rightarrow \lim_{n\to\infty} (s_n + t_n) = L + M.$$

2.7B Theorem: If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers, if $c \in R$, and if $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} c s_n = cL$.

To Prove; given $\epsilon > 0$ $\exists$ $N \in I$ such that

$$|cs_n - cL| < \epsilon \quad \forall n \geq N.$$

Proof given $\lim_{n\to\infty} s_n = L$, By defn, given $\epsilon > 0$ there exists $N \in I$ such that

$$|s_n - L| < \frac{\epsilon}{|c|} \quad \forall n \geq N \quad \text{— ①}$$

now $\forall n \geq N$

$$\begin{aligned} |cs_n - cL| &= |c(s_n - L)| \\ &\leq |c|\,|s_n - L| \\ &< |c| \frac{\epsilon}{|c|} \end{aligned}$$

$$|cs_n - cL| < \epsilon \quad \forall n \geq N$$

$$\Rightarrow \lim_{n\to\infty} c s_n = cL.$$

2.7C Theorem: a) If $0 < x < 1$, then $\{x^n\}_{n=1}^{\infty}$ converges to zero. b) If $1 < x < \infty$ then $\{x^n\}_{n=1}^{\infty}$ diverges to infinity.

Proof Given $0 < x < 1$ then $x^n > x^{n+1}$

(6) $x > x^2 > x^3 > \cdots x^n > x^{n+1} > \cdots$

$\therefore$ seq $\{x^n\}_{n=1}^{\infty}$ is decreasing sequence.

and $\forall n \in I$ $\quad x^n > 0 \Rightarrow$ Seq $\{x^n\}_{n=1}^{\infty}$ is bounded below

[Thm ~~We~~ know that decreasing sequence bounded below is convergent]

$\therefore \lim_{n\to\infty} x^n = L$ (say)

$\therefore \lim_{n\to\infty} x \cdot x^n = xL$

(e) $\lim_{n\to\infty} x^{n+1} = xL$ But seq $\{x^{n+1}\}_{n=1}^{\infty}$ is a Sub sequence of $\{x^n\}_{n=1}^{\infty}$

We know that Every Sub seq of a convergent sequence is converges to the same limit.

$\therefore xL = L$

$xL - L = 0$

$L(x-1) = 0$ But $0 < x < 1$

$\Rightarrow x - 1 \neq 0$

$\therefore L = 0$

$\therefore \lim_{n\to\infty} x^n = 0 \Rightarrow$ If $0 < x < 1$, then the Sequence $\{x^n\}_{n=1}^{\infty}$ converges to zero [Hence proof (a)]

Proof (b) If $1 < x < \infty$ To prove $\{x^n\}_{n=1}^{\infty}$ diverges to infinity

If $x > 1$ then $x < x^2 < x^3 < \cdots < x^n < x^{n+1} < \cdots$

$\Rightarrow$ Seq $\{x^n\}_{n=1}^{\infty}$ is increasing sequence

To seq $\{x^n\}^{\infty}$ is not bounded above.

Let us assume seq $\{x^n\}_{n=1}^{\infty}$ is bounded above (1)

[Increasing sequence bounded above is convergent]

$\therefore \lim_{n\to\infty} x^n = L$

$\lim_{n\to\infty} x \cdot x^n = xL$

$\Rightarrow \lim_{n\to\infty} x^{n+1} = xL$

but seq $\{x^{n+1}\}_{n=1}^{\infty}$ is Sub-seq of seq $\{x^n\}_{n=1}^{\infty}$

$\Rightarrow xL = \cancel{x}L$

$xL - \cancel{L} = 0$

$\cancel{x(L-1) = 0}$

$L(x-1) = 0$

~~But~~ $\cancel{x}$

But $x > 1$

$x - 1 > 0$

$x - 1 \neq 0$

$\Rightarrow L = 0$

$\Rightarrow$ The seq $\{x^n\}_{n=1}^{\infty}$ converges to Zero

which is Contradiction / $\because x > 1$ then $\{x^n\}_{n=1}^{\infty}$ never converges to Zero.

$\therefore$ seq $\{x^n\}_{n=1}^{\infty}$ is not bounded above.

$\therefore$ We know that Increasing seq and not bounded above is diverges to infinity.

$\therefore$ when $1 < x < \infty$ the sequence $\{x^n\}_{n=1}^{\infty}$ diverges to ∞.